Non-Standard Solutions to the Euler System of Isentropic Gas Dynamics

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

ELISABETTA CHIODAROLI

aus

Italien

Promotionskomitee

Prof. Dr. Camillo De Lellis (Vorsitz)
Prof. Dr. Thomas Kappeler

Zürich, 2013

Abstract

This thesis aims at shining some new light on the *terra incognita* of multi-dimensional hyperbolic systems of conservation laws by means of techniques whose application to this field is a brand new idea. In particular, our attention focuses on the isentropic compressible Euler equations of gas dynamics, the oldest but yet most prominent paradigm for this class of equations. The theory of the Cauchy problem for hyperbolic systems of conservation laws in more than one space dimension is still in its dawning and has been facing some basic issues so far: do there exist weak solutions for any initial data? how to prove well-posedness for weak solutions? which is a good space for a well-posedness theory? are entropy inequalities good selection criteria for uniqueness? Inspired by these interesting questions, we obtained some new results here collected. First, we present a counterexample to the well-posedness of entropy solutions to the Cauchy problem for the multi-dimensional compressible Euler equations: in our construction the entropy condition is not sufficient as a selection criterion for unique solutions. Furthermore, we show that such a non-uniqueness theorem holds also for some Lipschitz initial data in two space dimensions. Our results and constructions build upon the method of convex integration developed by De Lellis-Székelyhidi [**DLS09, DLS10**] for the incompressible Euler equations and based on a revisited "h-principle".

Finally, we prove existence of weak solutions to the Cauchy problem for the isentropic compressible Euler equations in the particular case of regular initial density. This result indicates the way towards a more general existence theorem for generic initial data. The proof ultimately relies once more on the methods developed by De Lellis and Székelyhidi in [**DLS09**]-[**DLS10**].

Zusammenfassung

Diese Doktorarbeit beabsichtigt, neues Licht auf die *terra incognita* der mehrdimensionalen hyperbolischen Systeme von Erhaltungsgleichungen, mit Hilfe für dieses Gebiet neuer Techniken, zu werfen. Unser Interesse konzentriert sich insbesondere auf die isentropen kompressiblen Euler Gleichungen der Gasdynamik, die das älteste und doch prominenteste Beispiel für diese Klasse von Gleichungen sind. Die Theorie des Cauchy Problems für hyperbolische Systeme von Erhaltungsgleichungen in mehr als einer Raumdimension befindet sich noch im Anfangsstadium ihrer Entwicklung und stellt sich bisher folgenden grundlegenden Fragestellungen: Existieren schwache Lösungen für alle Anfangsdaten? Wie ist die "well–posedness" für schwache Lösungen zu beweisen? Welche Räume bilden eine gute Basis für die "well–posedness" Theorie? Sind Entropieungleichungen gute Auswahlkriterien für die Eindeutigkeit?

Inspiriert durch diese interessanten Fragen, haben wir einige neue Resultate, die hier gesammelt sind, erhalten. Zuerst präsentieren wir ein Gegenbeispiel zur "well–posedness" der Entropielösungen mehrdimensionaler kompressibler Euler Gleichungen: In unserer Konstruktion ist die Entropiekondition als Selektionskriterium für eindeutige Lösungen nicht hinreichend. Ausserdem zeigen wir, dass ein solcher nicht-Eindeutigkeitssatz auch für ein Lipschitz Anfangsdatum in zwei Raumdimensionen gilt. Unsere Ergebnisse und Konstruktionen basieren auf der Methode der konvexen Integration, die von De Lellis und Székelyhidi [**DLS09**]-[**DLS10**] für die inkompressiblen Euler Gleichungen entwickelt wurde und auf einem modifizierten "h-Prinzip" basiert.

Des Weiteren beweisen wir die Existenz schwacher Lösungen des Cauchy-Problems für die isentropen kompressiblen Euler Gleichungen im Spezialfall regelmässiger Anfangsdichte. Dieses Ergebnis ebnet den Weg zu einem allgemeinen Existenzsatz für generische Anfangsdaten. Der Beweis baut einmal mehr auf den Methoden von De Lellis und Székelyhidi in [**DLS09**]–[**DLS10**] auf.

Acknowledgements

My first thank goes to my advisor Prof. Dr. Camillo De Lellis for being a guide and a great font of mathematical inspiration. In him I found an example of inextinguishable enthusiasm, ideas and taste for beautiful mathematics. I thank him for all the opportunities he offered me during my PhD.

I am grateful to my colleagues and friends, who rendered the years of my PhD a pleasant experience of life and friendship.

My sincere gratitude goes to my family for their continuous and loving support and encouragement. Finally, I would like to express a special thank to Lorenzo for being always at my side.

Elisabetta Chiodaroli, January 2013.

Contents

Introduction

The main topic of this thesis is the study of the compressible Euler equations of isentropic gas dynamics

$$
\begin{cases}
\partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\
\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] \;=\; 0 \\
\rho(0,\cdot) \;=\; \rho^0 \\
v(0,\cdot) \;=\; v^0
\end{cases}
\tag{0.1}
$$

whose unknowns are the density ρ and the velocity v of the gas, while p is the pressure which depends on the density ρ. In particular, we are concerned with the Cauchy problem (0.1) and with its possible (or not) well-posedness theory.

The isentropic compressible Euler equations (0.1) are an archetype for systems of hyperbolic conservation laws. Conservation laws model situations in which the change of amount of a physical quantity in some domain is due only to an income or an outcome of that quantity across the boundary of the domain. Indeed, this is the case also for system (0.1), where the equations involved state the balance laws for mass and for linear momentum.

The apparent simplicity of conservation laws, and in particular of system (0.1), contrasts with the difficulties encountered when solving the Cauchy problem. To illustrate the mathematical difficulties, let us say that there has not been so far a satisfactory result concerning the global existence of a solution to the Cauchy problem. In the context of classical solutions the Cauchy problem is known to be only locally well-posed: the resulting smooth solutions are stable, even within the broader class of weak solutions, but their life span is finite (see [**Daf10**, Theorems 5.1.1 and 5.3.1]). The Cauchy problem in the large may be considered only in the context of weak solutions. The well-posedness theory for weak solutions of hyperbolic conservation laws is presently understood only in the scalar case (one equation) thanks to the seminal work of Kruzkov [**Kru70**], and in the one–dimensional

case (one space–dimension) via the Glimm scheme [**Gli65**] or the more recent vanishing–viscosity method of Bianchini and Bressan [**Bre95**] and [**BB05**]. On the contrary, the general case is very far from being understood.
For this reason, a wise approach is to tackle some particular examples, in hope of getting some general insight.

This motivates our interest on the paradigmatic system of conservation laws (0.1). On the one hand, we can obtain a partial result on the existence of weak solutions to (0.1) for general initial momenta and regular initial density; on the other hand, building upon the same methods (see [**DLS09**]-[**DLS10**]), we can prove non–uniqueness for entropy solutions of (0.1) even for Lipschitz initial data. Our conclusions represent a step forward in the understanding of multi-dimensional hyperbolic systems of conservation laws, but at the same time they raise new and intriguing open questions.

In this introductory chapter, we frame our dissertation presenting an overview of the theory of hyperbolic systems of conservation laws and highlighting open problems and challenges of the subject. Finally, we will present the main results contained in this thesis and we will outline its structure.

0.1. Hyperbolic systems of conservation laws

Hyperbolic systems of conservation laws are systems of partial differential equations of evolutionary type which arise in several problems of continuum mechanics. One of their characteristics is the appearance of singularities (known as *shocks*) even starting from smooth initial data. In the last decades a very successful theory has been developed in one–space dimension but little is known about the general Cauchy problem in more than one–space dimension after the appearance of singularities. Recently, building on some new advances on the theory of transport equations, well-posedness for a particular class of systems has been proved. On the other hand, introducing techniques which are completely new in this context, it has been possible to establish an ill-posedness result for bounded entropy solutions of the Euler system of isentropic gas dynamics (0.1). Connected to these recent advances, there have been various open questions: how to conjecture well-posedness for general systems of conservation laws in several space dimensions? in which functional space? what structural properties of the Euler system of isentropic gas dynamics underlie the mentioned ill-posedness result? and in which class of initial data does this result hold? This thesis was inspired by such challenging questions and attempted to move some steps forward in the process of answering them.

0.1.1. Survey on the classical theory. The theory of nonlinear hyperbolic systems of conservation laws traces its origins to the mid 19th century and has developed over the years conjointly with continuum physics. The great number of books on the theoretical and numerical analysis published in recent years is an evidence of the vitality of the field. But, what does the denomination "hyperbolic systems of conservation laws" encode? They are systems of nonlinear, divergence structure first-order partial differential equations of evolutionary type, which are typically meant to model *balance laws*. In fact, the vast majority of noteworthy hyperbolic systems of conservation laws came up in physics, where differential equations were derived from corresponding statements of balance of an extensive physical quantity coupled with constitutive relations for a material body (see for instance [**Daf10**] and [**Ser99**]). In the most general framework, the field equation resulting from this *coupling process* reads as

$$\partial_t U + \mathrm{div}_x[F(U)] = 0 \tag{0.2}$$

where the unknown is a vector valued function

$$U = U(t,x) = (U^1(t,x), ..., U^k(t,x)) \quad ((t,x) \in \Omega \subset \mathbb{R}_t \times \mathbb{R}^m_x), \tag{0.3}$$

the components of which are the densities of some conserved variables in the physical system under investigation, while the *flux* function F controls the rate of loss or increase of U through the spatial boundary and satisfies suitable "hyperbolicity conditions", namely that for every fixed U and $\nu \in S^{m-1}$, the $k \times k$ matrix

$$\sum_{\alpha=1}^{m} \nu_\alpha \mathrm{D}F_\alpha(U)$$

has real eigenvalues and k linearly independent eigenvectors.

Solutions to hyperbolic conservation laws may be visualized as propagating waves. When the system is nonlinear, the profile of compression waves gets progressively steeper and eventually breaks, generating jump discontinuities which propagate on as *shocks*. This behavior is demonstrated by the simplest example of a nonlinear hyperbolic conservation law in one space variable, namely the *Burgers equation*

$$\partial_t U(t,x) + \partial_x \left(\frac{1}{2} U^2(t,x) \right) = 0. \tag{0.4}$$

The appearance of singularities, even when starting from regular initial data, drives the theory to deal with weak solutions. This difficulty is compounded further by the fact that, in the context of weak solutions, uniqueness is lost. To see this, one can consider the Cauchy problem for the Burgers equation (0.4), with initial data

$$u(0,x) = \begin{cases} -\ \ 1, & x < 0 \\ \ \ \ \ 1, & x > 0. \end{cases} \tag{0.5}$$

The problem (0.4), (0.5) admits infinitely many solutions, including the family

$$u_\beta(t,x) = \begin{cases} -\ \ 1, & -\infty < x \leq -t \\ \ \ \ \ \frac{x}{t}, & -t < x \leq -\beta t \\ -\ \ \beta, & -\beta t < x \leq 0 \\ \ \ \ \ \beta, & 0 < x \leq \beta t \\ \ \ \ \ \frac{x}{t}, & \beta t < x \leq t \\ \ \ \ \ 1, & x > 0, \end{cases}$$

for any $\beta \in [0,1]$.

It thus becomes necessary to devise proper criteria for weeding out unstable, physically irrelevant, or otherwise undesirable solutions, in hope of singling out admissible weak solutions. The issue of admissibility of weak solutions to hyperbolic systems of conservation laws is a central question of the theory and stirred up a debate quite early in the development of the subject. Continuum physics naturally induces such admissibility criteria through the Second Law of Thermodynamics. These may be incorporated in the analytical theory, either directly, by stipulating outright that admissible solutions should satisfy "entropy" inequalities, or indirectly, by equipping the system with a minute amount of diffusion, which has negligible effect on smooth solutions but reacts stiffly in the presence of shocks, weeding out those that are not thermodynamically admissible. In the framework of the general theory of hyperbolic systems of conservation laws, the use of entropy inequalities to characterize admissible solutions was first proposed by

Kruzkov [**Kru70**] and then elaborated by Lax [**Lax71**]. The idea of regarding inviscid gases as viscous gases with vanishingly small viscosity is quite old; there are hints even in the seminal paper by Stokes [**Sto48**]. The important contributions of Rankine [**Ran70**], Hugoniot [**Hug89**] and Rayleigh [**Ray10**] helped to clarify the issue.

From the standpoint of analysis, a very elegant, definitive theory is available for the case of scalar conservation laws (i.e. when $k = 1$ in (0.3)), in one or several space dimensions. The special feature that sets the scalar balance law apart from systems of more than one equation is the size of its family of entropies: in the scalar case the abundance of entropies induces an effective characterization of admissible weak solutions as well as very strong L^1-stability and L^∞-monotonicity properties. Armed with such powerful a priori estimates, one can construct admissible solutions in a number of ways. In the one-dimensional case the qualitative theory was first developed in the 1950's by the Russian school, headed by Oleinik [**Ole54, Ole57, Ole59**], while the first existence proof in several space dimensions was established a few years later by Conway and Smoller [**CS96**], who recognized the relevance of the space BV. The definitive treatment in the space BV was later given by Volpert [**Vol67**]; building on Volpert's work, Kruzkov [**Kru70**] proved the well-posedness for admissible weak solutions. As a consequence of Kruzkov's results when the initial data are functions of locally bounded variation then so are the solutions. Remarkably, even solutions that are merely in L^∞ exhibit the same geometric structure as BV functions, with jump discontinuities assembling on "manifolds" of codimension one (see [**DLOW03**] and [**DLR03**])

By contrast, when dealing with systems of conservation laws, it is still a challenging mathematical problem to develop a theory of well-posedness for the Cauchy problem of (0.2) which includes the formation and evolution of shock waves. In one space dimension, namely when $m = 1$ in (0.3), this problem has found recently a quite satisfactory and general answer, thanks to the efforts of generations of mathematicians: the general mathematical framework of the theory was set in the seminal paper of Lax [**Lax57**]; the first existence result (for small BV data) is due to Glimm [**Gli65**] in the sixties; Bressan [**Bre95**] (see also [**Bre00**]) finally proved well-posedness of the Cauchy problem and later Bianchini and Bressan [**BB05**] showed convergence of viscosity solutions to the the unique entropy weak solution. The higher dimensional case is terra incognita: how to conjecture stability is still an open problem. Indeed, in several space dimensions, the situation is clearly less favourable: the success of the spaces L^∞ and BV in one space dimension is due to the fact that they are algebras allowing for the treatment of the rather strong non–linearity of the equations; however, the works of Brenner [**Bre66**] and Rauch [**Rau86**], which are concerned with linear systems, show that these spaces cannot be adapted to the multi–dimensional case. We are thus in presence of a paradox which has up to the present not been resolved: *to find a function space which is an algebra, probably constructed on L^2 and which contains enough discontinuous solutions*. Moreover, even a general existence result for weak solutions in more than one space dimension is missing so far. The theory is in its infancy.

0.1.2. Recent results. Recently, some attention was devoted to a first "toy example" falling in the class (0.2). This system, called Keyfitz–Kranzer system, is clearly very peculiar and, compared to the most relevant systems coming from the physical literature, has many more features. It reads as

$$\begin{cases} \partial_t u + \sum_{j=1}^m \frac{\partial}{\partial x_j}(f^j(|u|)u) \ = \ 0 \\ u(0,\cdot) \ = \ u^0 \end{cases} \tag{0.6}$$

where for any $j = 1, \dots, m$ the map $f^j : \mathbb{R}^+ \to \mathbb{R}$ is assumed to be smooth. In this case the non–linearity depends only on the modulus of the solutions. Most notably the system (0.6) decouples into a nonlinear conservation law for the modulus of $\rho := |u|$

$$\partial_t \rho + \mathrm{div_x}(\mathrm{f}(\rho)\rho) \ = \ 0$$

and a system of linear transport equations for the angular part $\theta := u/|u|$

$$\partial_t \theta + f(\rho) \cdot \nabla_x \theta \ = \ 0.$$

However, it does develop singularities in finite time and a theory of well–posedness of singular solutions was still lacking up to few years ago. Thanks to a groundbreaking paper of Ambrosio (see [**Amb04**]), it was possible to solve this problem in a very general and satisfactory way (see [**ADL03**]): well posedness of *renormalized entropy solutions* in the class of maps $u \in L^\infty([0,T] \times \mathbb{R}^m; \mathbb{R}^k)$ with $|u| \, inBV_{loc}$ has been proven by Ambrosio, Bouchut and De Lellis in [**ABDL04**]. Moreover, the problem has been used to show that, even in this very particular case, there is no hope of getting estimates in some of the classical function spaces which are used in the one-dimensional theory.

In a recent work De Lellis and Székelyhidi found striking counterexamples to the well-posedness of bounded entropy solutions to the isentropic system of gas-dynamics (0.1) (see [**DLS10**]). These examples build on a previous work ([**DLS09**]) where they introduced techniques from the theory of differential inclusions to construct very irregular solutions to the incompressible Euler equations. The isentropic system of gas dynamics in Eulerian coordinates (0.1) is the oldest hyperbolic system of conservation laws. The hyperbolicity condition for system (0.1) reduces to the monotonicity of the pressure as a function of the density: $p'(\rho) > 0$. In this thesis the pressure p will always satisfy this assumption. Weak solutions of (0.1) are bounded functions which solve the system in the sense of distributions. Admissible (or entropy) solutions can be characterized as those weak solutions which satisfy an additional inequality, coming from the conservation law for the energy of the system. In the paper [**DLS10**], De Lellis and Székelyhidi show L^∞ initial data, with strictly positive piecewise constant density, which allow for infinitely many admissible solutions of (0.1) in more than one space dimension, all with strictly positive density:

THEOREM 0.1.1 (Theorem from [**DLS10**]). *Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) of (0.1) with $\rho \geq c > 0$.*

This result proves that the space L^∞ is ill-suited for well-posedness of entropy solutions; moreover it makes us believe that admissibility inequalities are not the "right" selection criteria. Of course, from Theorem 0.1.1 arises a cascade of questions which point in many grey areas; some of these open questions are at the core of this dissertation.

0.1.3. Motivating problems. In this paragraph, we summarize the main issues which motivated the research presented in this thesis. They do not exhaust the immeasurable amount of open problems in the field of hyperbolic systems of conservation laws but they are indicative of the topicality of this branch of mathematics. Moreover, as illustrated by the following motivating questions, our concern is not only for the novelty of the results such questions could lead to, but also for the techniques involved.

- *Existence results*: the lack of satisfactory existence results for weak solution of multi–dimensional systems of conservation laws is a glaring symptom of the difficulties underlying the theory. In particular, for bounded initial data, but of arbitrary size, only 2×2 systems in one–space dimension have been tackled by the method of *compensated compactness*. Is it possible to prove some existence result at least for the particular system of isentropic gas dynamics?

- *Well-posedness of admissible solutions*: how to separate the wheat from the chaff, the solutions observed in nature (the *physically admissible* ones) from those that are only mathematical artefacts is one of the central questions in the theory of multi–dimensional systems of conservation laws. In particular, are entropy (admissibility) inequalities efficient as selection criteria? Theorem 0.1.1 seems to give a negative answer to the aforementioned question. Should this inefficiency of entropy inequalities believed to be a "universal law"?

- *Ill-posedness for the isentropic Euler system*: the surprising result 0.1.1 from [**DLS10**] left unsolved the question whether the system (0.1) directly allows for the construction of the paper [**DLS10**]. Such an issue is connected not only to the efficiency of admissible criteria to weed out non–physical solutions, but also to a development of the techniques on which [**DLS10**] bases.

- *Ill-initial data*: relying only on [**DLS10**] one could argue that phenomena of ill-posedness could be restricted to very particular initial data and that for a large class of them, one could hope for a uniqueness theorem. We aimed at understanding better for which initial data such constructions are possible. In particular we questioned the case of Riemann initial data.

- *Further applications*: the new method introduced in [**DLS09**] has been extended in a quite direct way to the isentropic system of gas-dynamics. Another interesting question concerns possible extensions and further applications of the idea coming from [**DLS10**]; for instance could it be applied to other systems of conservation laws?

- *Suitable functional spaces*: The inadequacy of the spaces L^∞ or BV for multi–dimensional systems of conservation laws raises the issue of finding another functional space for the study of weak solutions. But no satisfactory space has been suggested until now.

0.2. Main results and outline of the thesis

This thesis consists of five chapters whose content we are going to disclose.

The results here presented represent new developments and applications of the innovative approach introduced by De Lellis and Székelyhidi in [**DLS09**]-[**DLS11**]. Their methods brought in the realm of fluid dynamics techniques coming from Gromov's convex integration [**Gro86**] and strategies from the theory for differential inclusions [**DM97**], [**KMS03**] and remarkably combined these tools to construct non-standard solutions to the incompressible Euler equations and to the compressible ones as well. Since our achievements strongly rely on this new approach, we devote Chapter 1 to a brief compendium on the related background theory. In Chapter 1 we aim at introducing the reader to the interesting theory lying behind Theorem 0.1.1. In particular, we will explain how Gromov's work on partial differential relations and on convex integration together with Kirchheim, Müller and Sverák's approach to study the properties of nonlinear partial differential equations can concur to construct solutions to equations from fluid dynamics. We will also hint the analogies between problems in differential geometry, where the idea of convex integration originally arose, and the incompressible Euler equations. Finally, we will give an overview of the recent results in fluid dynamics obtained as further advancements of De Lellis and Székelyhidi's ideas.

The Introduction together with the first chapter provides a preface to the core of the thesis: indeed, in the subsequent chapters, the tools introduced in Chapter 1 will be applied to the compressible Euler equations (0.1) allowing for new results.

Chapter 2 contains the first important theorem of the thesis:

THEOREM 0.2.1 (Non-uniqueness of entropy solutions with arbitrary density). *Let $n \geq 2$. Then, for every initial density $\rho^0 \in C^1$ with $\rho^0 \geq c > 0$ and for any given function p, there exist an initial velocity $v^0 \in L^\infty$ and a time $T > 0$ such that there are infinitely many bounded admissible solutions (ρ, v) of (0.1) on $\mathbb{R}^n \times [0, T[$, all with density bounded away from 0.*

This theorem is an improvement of Theorem 0.1.1: using the same techniques as in Theorem 0.1.1, we can show that the same non-uniqueness result holds for any choice of the initial density (see also [**Chi11**]). This highlights that the main role in the loss of uniqueness is due to the velocity field. While the proof of Theorem 0.1.1 relies on a non-uniqueness result for the incompressible Euler equations, and hence yields "piecewise incompressible" solutions, Theorem 0.2.1 is achieved by applying directly to (0.1) De Lellis and Székelyhidi's ideas. Yet, the solutions constructed in Theorem 0.2.1 allow for wild oscillations only in the velocity. The general case which would include wild oscillations in the density as well is presently under investigation (cf. [**CDLK**]).

The initial data v^0 as in Theorem 0.1.1 were extremely irregular and left open the question whether the ill–posedness was not due to the irregularity of the data, rather than to the irregularity of the solutions. Theorem 0.2.1 provides a first answer showing that data with very regular initial density but irregular velocities still allow for non–uniqueness of admissible solutions. A surprising corollary of De Lellis and Székelyhidi's result on the incompressible Euler equations is that in two space dimensions even for some smooth initial data non–uniqueness of bounded admissible solutions arises after the first blow–up time (see [**CDL**]):

THEOREM 0.2.2 (Ill–posedness with Lipschitz data). *There are Lipschitz initial data (ρ^0, v^0) for which there are infinitely many bounded* *admissible solutions* (ρ, v) *of* (0.1) *on* $\mathbb{R}^2 \times [0, \infty[$ *with* $\inf \rho > 0$. *These solutions are all locally Lipschitz on a finite interval of time where they therefore all coincide with the unique classical solution.*

Theorem 0.2.2 entails that the classical entropy inequality does not ensure uniqueness of the solutions even for Lipschitz initial data. It remains however an open question how irregular the solutions have to be in order to display the pathological behaviour of Theorem 0.2.2. One could speculate that, in analogy to what has been shown recently for the incompressible Euler equations, even a "piecewise Hölder regularity" might not be enough; see [**DLS12**], [**DLSZ12**], [**BDS13**] and in particular [**DA13**].

The proof of Theorem 0.2.2 is the content of Chapter 3. It is achieved by showing the existence of classical Riemann data (i.e. pure jump discontinuities across a line) which can be generated by a compression wave and for which there are infinitely many bounded admissible solutions of (0.1). Indeed Theorem 0.2.2 is obtained in Chapter 3 as a Corollary of the following main result:

THEOREM 0.2.3 (Non–standard solutions with Riemann data). *Assume* $p(\rho) = \rho^2$. *Then there are Riemann data for which there are infinitely many bounded admissible solutions* (ρ, v) *of* (0.1) *on* $\mathbb{R}^2 \times [0, \infty[$ *with* $\inf \rho > 0$. *These data are indeed all generated by classical compression waves.*

Chapter 4 represents a complement to Chapter 3. In Chapter 4 we restrict our attention to the 1-*dimensional Riemann problem* for the compressible Euler equations with the same choice of Riemann data as in Theorem 0.2.3 (i.e. Riemann data allowing for the proof of Theorem 0.2.2): we show that such a problem admits unique self-similar solutions. This follows from classical considerations but since we have not been able to find a precise and complete reference, we include all the arguments here. Theorem 0.2.3 shows that as soon as the self–similarity assumption runs out, uniqueness is lost.

In Chapter 5 we address the issue of global existence of weak solution. Building upon the results of Chapter 2 we are able to prove the following Theorem concerning the system (0.1).

THEOREM 0.2.4 (Existence of weak solution with arbitrary initial momentum). *Let* $\rho^0 \in C^1$ *and* $v^0 \in L^2$ *such that* $\operatorname{div} v^0 = 0$. *Then there exists a global weak solution* (ρ, v) *(in fact, infinitely many) of the Cauchy problem for the compressible Euler equations* (0.1) *with initial data* (ρ^0, v^0).

Theorem 0.2.4 is just a first step towards the most desirable result of existence of weak solutions of (0.1) starting out from any given bounded initial density and momentum. Such an outcome would be of great impact, since so far no existence result for weak solutions of multi-dimensional hyperbolic systems of conservation laws with generic initial data is available. Of course Theorem 0.2.4 is able to deal only with regular initial densities, but the more general result is believed to hold building on the improvement of Theorem 0.2.1 (see [**CDLK**]).

The thesis is conceived in such a way that every chapter can be read both as the continuation of what precedes or independently.

CHAPTER 1

The h–principle and convex integration

The *h-principle* is an umbrella–concept forged by Gromov in the 1960s and 1970s ([**Gro70**]) to unify a series of counterintuitive results in topology and differential geometry. This principle is a strong property characterizing the set of solutions of *differential relations*: a differential relation is *soft* or *abides by the h-principle* if its solvability can be determined on the basis of purely homotopic calculus. By *differential relation* we mean a constraint on maps between two manifolds and on their derivatives as well. PDEs are examples of differential relations. It is striking that many differential relations, mostly rooted in differential geometry and topology, are soft. Two famous examples of the softness phenomenon are the Nash-Kuiper C^1 isometric embedding theory and the Smale's sphere eversion.

Why are we interested in the h–principle? It is surprising how the results on the isentropic Euler equations of gas dynamics presented in this thesis are based on a revisited h-principle. This new variant of h-principle has been first devised by De Lellis and Székelyhidi for the incompressible Euler equations (see [**DLS09**]) and lead to new developments for several equations in fluid dynamics as the ones of this thesis. Indeed, even if the original h-principle of Gromov pertains to various problems in differential geometry, De Lellis and Székelyhidi showed in their groundbreaking paper [**DLS09**] that the same principle and similar methods could be applied to problems in mathematical physics. The work by De Lellis and Székelyhidi found its breeding ground in the important paper by Müller and Šverák [**MS03**], where they extended the method of convex integration (introduced by Gromov to prove the h-principle) to Lipschitz mappings and noticed the strong connections between the existence theory for differential inclusions and the h-principle.

We do not pretend here to give an account of the extremely wide literature on this topic, but we rather prefer to illustrate some specific instances of the h-principle as a jumping off point for a general understanding of the subject. In particular, we will spend some words on the method of convex integration which will be recalled in the next chapters for the arguments of our constructions.

1.1. Partial differential relations and Gromov's h-principle

In this section we will introduce the idea behind the h-principle by illustrating Gromov's original formalism.

A *partial differential relation* $\mathcal{R}$ is any condition imposed on the partial derivatives of an unknwon function. A solution of $\mathcal{R}$ is any function which satisfies this relation. Any differential relation has an underlying algebraic relation which one gets by substituting derivatives by new independent variables. A simple example of differential relations are ordinary differential equations or inequalities. We can consider, for instance, the differential equation $y'(x) = y^2(x)$; then the underlying algebraic relation is obtained by introducing the new variable z in place of the derivative y': the resulting relation is simply $z = y^2$ seen as a constraint in $\mathbb{R}^3$ with coordinates (x, y, z). In this language, a solution of the corresponding algebraic relation is called a *formal solution* of the original differential relation $\mathcal{R}$. The difference between genuine and formal solutions in this specific example becomes clear as soon as we interpret genuine solutions as functions (in fact sections) $f : \mathbb{R} \to \mathbb{R}^3$, $f(x) = (x, y(x), y'(x))$ with $y'(x) = y^2(x)$ (this amounts to using the language of jets which we do not want to get into herein). Clearly, the existence of a formal solution is a necessary condition for the solvability of a differential relation $\mathcal{R}$. In the previous example, formal solutions are functions $g : \mathbb{R} \to \mathbb{R}^3$, $g(x) = (x, y(x), z(x))$ with $z(x) = y^2(x)$. The philosophy behind the h-principle consists in the following: before trying to solve $\mathcal{R}$ one should check whether $\mathcal{R}$ admits a formal solution. The problem of finding formal solutions is of purely homotopy-theoretical nature. It could seem, at first thought, that existence of a formal solution cannot be sufficient for the genuine solvability of $\mathcal{R}$. Indeed, finding a formal solution is an algebraic problem which is a dramatical simplification of the original differential problem. Thus it came as a big surprise when it was discovered in the second half of the twentieth century that there exist large and geometrically interesting classes of differential relations for which the solvability of the formal problem is sufficient for genuine solvability. Moreover, for many of these relations the spaces of formal and genuine solutions turned out to be much more closely related than one could expect. This property was formalized by Gromov [**Gro86**] as the following:

- **Homotopy principle (h-principle)**. A differential relation $\mathcal{R}$ satisfies the h-principle, or the h-principle holds for solutions of $\mathcal{R}$, if every formal solution of $\mathcal{R}$ is homotopic to a genuine solution of $\mathcal{R}$ through a homotopy of formal solutions.

The term "h-principle" was introduced and popularized by M. Gromov in his book [**Gro86**]. It is now clear that the h-principle does not hold for the differential equation $y'(x) = y^2(x)$ (we consider global solutions), while we could prove the h-principle for the equation $y'(x) = y(x)$ since every formal solution $f(x) = (x, y(x), y(x))$ can be joined via a homotopy H_t of formal solutions to the genuine solution $h(x) = (x, \exp(x), \exp(x))$ simply choosing $H_t(x) = (1-t)f(x) + th(x)$. These examples are of course trivial and not typical of situations where the h-principle is useful. In fact, the h-principle is rather useless in the classical theory of (ordinary or partial) differential equations because there it fails or holds for some trivial, or at least well known reasons, as in the above examples.

By contrast, for many differential relations rooted in topology and geometry the notion of h-principle appeared to be fundamental. There are several amazing unexpected cases in which the h-principles holds. A particular problem which abides by the h-principle can also be called *soft*. As already mentioned, the softness phenomena was first discovered in the fifties by Nash [**Nas54**] for isometric C^1-immersions and by Smale [**Sma58**]-[**Sma59**] for differential immersions. However, instances of the soft problems appeared earlier. In his dissertation and later in his book [**Gro86**], Gromov transformed Smale's and Nash's ideas into two powerful methods for solving partial differential relations: *continuous sheaves* method and the *convex integration* method. In the next section we will give an overview on Nash's construction, where the "spirit" of convex integration originally arose.

In the language pertaining to Gromov, the idea lying behind convex integration can be illustrated through an easy example which is suggested in [**EM02**]. Let us call a path

$$r : I = [0,1] \to \mathbb{R}^2,\ r(t) := (x(t), y(t)),$$

short if $x'(t)^2 + y'(t)^2 < 1$ a.e. $t \in I$. The inequality defining a short path is nothing else that a particular instance of partial differential relation. It easy to prove that any short path can be C^0-approximated by a solution of the equation $x'(t)^2 + y'(t)^2 = 1$ a.e. $t \in I$, which is another differential relation. This implies that the space of solutions $I \to \mathbb{R}^2$ of the differential equation $x'(t)^2 + y'(t)^2 = 1$ is C^0-dense in the space of solutions of the differential inequality $x'(t)^2 + y'(t)^2 < 1$. This elementary example illuminates the following idea at the core of convex integration: given a first order differential relation for maps $I \to \mathbb{R}^q$, it is useful to consider a "relaxed" differential relation which is the pointwise convex hull of the original relation

1.2. The *h*-principle for isometric embeddings

1.2.1. Isometric embedding problem. Let M^n be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. An *isometric immersion* of (M^n, g) into $\mathbb{R}^m$ is a map $u \in C^1(M^n; \mathbb{R}^m)$ such that the induced metric agrees with g. In local coordinates this amounts to the system

$$\partial_i u \cdot \partial_j u = g_{ij} \tag{1.1}$$

consisting of $n(n+1)/2$ equations in m unknowns. If in addition u is injective, it is an *isometric embedding*. Analogously, one defines a *short embedding* as a map $u : M^n \to \mathbb{R}^m$ such that the metric induced on M by u is shorter than g. In coordinates this translates into $(\partial_i u \cdot \partial_j u) \leq (g_{ij})$ in the sense of quadratic forms. Geometrically being short means that the embedding shrinks the length of curves. Equally, being isometric means that the length of curves is preserved.

The well-known result of Nash and Kuiper says that any short embedding in codimension one can be uniformly approximated by C^1 isometric embeddings.

THEOREM 1.2.1 (Nash-Kuiper theorem). *If $m \geq n+1$, then any short embedding can be uniformly approximated by isometric embeddings of class C^1.*

Note that Theorem 1.2.1 is not merely an existence theorem, but it shows that there exists a huge (essentially C^0-dense) set of solutions. Such a density of solutions is reminiscent of the example on short paths presented in the previous section. This type of abundance of solutions is a central aspect of Gromov's h-principle, for which the isometric embedding problem is a primary example. Indeed, we could ask whether there exists a regular homotopy $f_t : S^2 \to \mathbb{R}^3$ which begins with the inclusion f_0 of the unit sphere and ends with an isometric immersion f_1 into the ball of radius 1/2. One of the many counterintuitive implications of Nash and Kuiper's theorem is that we can answer positively to this question in case of C^1-immersions: S^2 can be C^1 isometrically embedded into an arbitrarily small ε-ball in Euclidean 3-space (for small ε there is no such C^2). The h-principle for isometric embeddings is rather striking, especially when compared to the classical rigidity result concerning the Weyl problem: if (S^2, g) is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$, then u is uniquely determined up to a rigid motion. Thus it is clear that isometric immersions have a completely different qualitative behaviour at low and high regularity (i.e. below and above C^2).

The proof of Theorem 1.2.1 involves an iteration technique called *convex integration*.

1.2.2. Nash-Kuiper's general scheme. The general scheme of the construction upon which the main results of this thesis build are strongly inspired by the method of Nash and Kuiper. It is then interesting to sketch here the Nash-Kuiper scheme. For simplicity we assume g to be smooth. Let us set some notation: given an immersion $u : M^n \to \mathbb{R}^m$, we denote by $u^\sharp e$ the pullback of the standard Euclidean metric e through u, so that in local coordinates

$$(u^\sharp e)_{ij} = \partial_i u \cdot \partial_j u.$$

Moreover we define

$$n_* = \frac{n(n+1)}{2}.$$

A Riemannian metric g on $\mathbb{R}^n$ is said to be *primitive* if $g = \alpha(x)(dl)^2$, where $l = l(x)$ is a linear function on $\mathbb{R}^n$ and α is a non-negative function with compact support. A Riemannian metric g on a manifold M is called *primitive* if there exists a local parametrization $\phi : \mathbb{R}^n \to U \subset M$ such that supp $g \subset U$ and $\phi^\sharp g$ is a primitive metric on $\mathbb{R}^n$.

For the sake of clarity, we will give the ideas for the proof of the following simplified version of Theorem 1.2.1

THEOREM 1.2.2. *Let Ω be an open bounded subset of $\mathbb{R}^n$ equipped with the Riemannian metric g and let u be a smooth strictly short immersion of (Ω, g) into $(\mathbb{R}^m, e)$, with $m \geq n+2$. Then, for every $\varepsilon > 0$ there exists a C^1 isometric immersion $\widetilde{u} : \Omega \hookrightarrow \mathbb{R}^m$ such that $\|u - \widetilde{u}\|_{C^0(\Omega)} < \varepsilon$, i.e.*

- $\widetilde{u} \in C^1(\overline{\Omega})$;
- $\partial_i \widetilde{u} \cdot \partial_j \widetilde{u} = g_{ij}$ *in* Ω;
- $\|u - \widetilde{u}\|_{C^0(\overline{\Omega})} < \varepsilon$.

The proof of Theorem 1.2.2 is based on an iteration of *stages*, and each *stage* consists of several *steps* whose purpose we are going to unravel.
Starting from u one defines a first perturbation as follows

$$u_1(x) := u(x) + \frac{a(x)}{\lambda} \left(\sin(\lambda x \cdot \xi)\zeta(x) + \cos(\lambda x \cdot \xi)\eta(x) \right),$$

where $\lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and ζ, η are unit normal vectors to $u(\Omega)$, i.e.

- $\zeta \perp \eta$ and $|\zeta| = |\eta| = 1$;
- $\zeta \perp \partial_i u$ & $\eta \perp \partial_i u$ for $i = 1, \ldots, n$.

Let us note that the condition of isometry $\partial_i \widetilde{u} \cdot \partial_j \widetilde{u} = g_{ij}$ can be equivalently written in terms of the matrix differential $\nabla u = (\partial_j u^i)_{ij}$ as $\nabla \widetilde{u}^{\mathrm{T}} \nabla \widetilde{u} = g$. Now, by easy computations one obtains:

$$\begin{aligned} \nabla u_1(x) = \nabla u(x) & \\ & + a(x) \left(\cos(\lambda x \cdot \xi)\zeta(x) \otimes \xi - \sin(\lambda x \cdot \xi)\eta(x) \otimes \xi \right) \\ & + \mathrm{O}\left(\frac{1}{\lambda} \right), \end{aligned}$$

and hence

$$(\nabla u_1)^{\mathrm{T}} \nabla u_1 = (\nabla u)^{\mathrm{T}} \nabla u + a^2(x) \xi \otimes \xi + \mathrm{O}\left(\frac{1}{\lambda} \right).$$

Picture 1 gives a geometric intuition of the perturbation introduced in u_1 in the case $n = 1$.

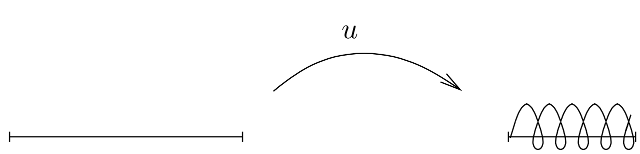

FIGURE 1. Geometric picture

Now, the purpose of a *stage* is to correct the error $g - u^\sharp e = u - (\nabla u)^{\mathrm{T}}\nabla u$. In order to achieve this correction, the error is locally decomposed into a sum of primitive metrics as follows

$$g - (\nabla u)^{\mathrm{T}}\nabla u \;=\; \sum_{k=1}^{n_*} a_k^2 \xi_k \otimes \xi_k \qquad \text{(locally)}.$$

Therefore, by iterating the procedure illustrated in the construction of u_1, i.e. by adding repeatedly "spirally perturbations" to u it should be possible to achieve u_N such that

- $(\nabla u_N)^{\mathrm{T}}\nabla u_N \;=\; g + \mathrm{O}\left(\frac{1}{\lambda}\right)$;
- $\|\nabla u_N - \nabla u\|_{C^0(\Omega)} = \sum_k \|a_k\|_{C^0(\Omega)} + \mathrm{O}\left(\frac{1}{\lambda}\right) \sim \; \left\|g - (\nabla u)^{\mathrm{T}}\nabla u\right\|_{C^0(\Omega)}^{1/2}$;
- $\|u_N - u\|_{C^0(\Omega)} \;=\; \mathrm{O}\left(\frac{1}{\lambda}\right)$.

Let us draw the attention to the fact that, when introducing a perturbation as in u_1, a, ζ and η may vary as x varies but that is not the case for ξ which is a fixed vector: this prevents us from correcting the error simply by taking the eigenvectors of $g(x) - \nabla u(x)^{\mathrm{T}}\nabla u(x)$. This involves the use of a "partition of unity" of the set of positive definite matrices, which we will not expound here. The previous considerations show which kind of estimates are involved when adding a primitive metric. Hence, the general Nash-Kuiper's scheme lies in the following iterations:

- *step*: a step involves adding one primitive metric; in other words the goal of a step is the metric change
$$u^\sharp e \to u^\sharp e + \sum a^2 \xi \otimes \xi;$$
- *stage*: a stage consists in decomposing the error into primitive metrics and adding them successively in steps.

The number of *steps* in a *stage* equals the number of primitive metrics in the above decomposition which interact. This equals n_* for the local construction and $(n+1)n_*$ for the global construction. Therefore iterating the estimates for one step over a single stage and then over the stages leads to the desired result.

1.2.3. Connection to the Euler equations. There is an interesting analogy between isometric immersions in low codimension and the incompressible and compressible Euler equations. In [**DLS09**] a method, which is very closely related to convex integration, was introduced to construct highly irregular energy-dissipating solutions of the incompressible Euler equations. In general the regularity of solutions obtained using convex integration agrees with the highest derivatives appearing in the equations. Being in conservation form, the "expected" regularity space for convex integration for the incompressible Euler equations should be C^0. In [**DLS09**] a weaker version of convex integration was applied, to produce solutions in L^∞ (see also [**DLS10**] for a slightly better space) and to show that a weak version of the h-principle holds (even if there is no homotopy there). Recently, De Lellis and Székelyhidi have proved the existence of continuous and even Hölder continuous solutions which dissipate the kinetic energy. Moreover the same method devised in [**DLS10**] led to new developments in fluid dynamics in particular for the Euler system of isentropic compressible gas dynamics. When comparing the Euler equations (both compressible and incompressible) and the Nash-Kuiper result, the reader should take into account that, in this analogy, the velocity field of the Euler equations corresponds to the differential of the embedding in the isometric embedding problem. All these aspects are surveyed in the note [**DLS11**].

The understanding of Nash's construction is in a way a starting point for the approach developed by De Lellis and Székelyhidi in [**DLS09**]. As in the case of Nash, the solution of the incompressible Euler equations is generated by an iteration scheme: at each stage of this iteration a subsolution is produced from the previous one by adding some special perturbations, which oscillate quite fast. Hence, the final result of the iteration scheme is the superposition of infinitely many perturbations which converge suitably to an exact solution.

1.3. The h-principle and the equations of fluid dynamics

In a recent note [**DLS11**], De Lellis and Székelyhidi disclosed the analogy between recent outcomes in fluid dynamics (included the ones presented in this thesis) and some h-principle-type results in differential geometry, as the previously presented Nash-Kuiper Theorem. More precisely, the survey by De Lellis and Székelyhidi aims at showing how the theorems in fluid mechanics represent a suitable variant of Gromov's h-principle. In this section, we will retrace the main points of [**DLS11**] so to place the results of this thesis in a more general context.

1.3.1. The general framework. Kirchheim, Müller and Šverák in [**KMS03**] outlined an approach to study the properties of nonlinear partial differential equations through the geometric properties of a set in the space of $m \times n$ matrices which is naturally associated to the equation. This approach draws heavily on Tartar's work on oscillations in nonlinear PDEs and compensated compactness and on Gromov's work on partial differential relations and convex integration.

What does this method consist of? Following Tartar's framework [**Tar79**], many nonlinear systems of PDEs for a map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$ can be naturally espressed as a combination of a linear system of PDEs of the form

$$\sum_{i=1}^{n} A_i \partial_i z = 0 \text{ in } \Omega \tag{1.2}$$

and a pointwise nonlinear constraint

$$z(x) \in K \text{ a.e. } x \in \Omega, \tag{1.3}$$

where

- $z : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$ is the unknown state variable;
- A_i are constant $m \times d$ matrices;
- $K \subset \mathbb{R}^d$ is a closed set.

Plane waves are solutions of (1.2) of the form

$$z(x) = ah(x \cdot \xi), \tag{1.4}$$

where $h : \mathbb{R} \to \mathbb{R}$. Then, one defines the *wave cone* Λ related to one–dimensional solutions and given by the states $a \in \mathbb{R}^d$ such that for any choice of the profile h the function (1.4) solves (1.2):

$$\Lambda := \left\{ a \in \mathbb{R}^d : \; \exists \xi \in \mathbb{R}^n \setminus \{0\} \text{ with } \sum_{i=1}^{n} \xi_i A_i a = 0 \right\}. \tag{1.5}$$

Equivalently Λ characterizes the directions of one–dimensional oscillations compatible with (1.2). Given a cone Λ we say that a set S is lamination convex (with respect to Λ) if for any two points $A, B \in S$ with $B - A \in \Lambda$ the whole segment $[A, B]$ belongs to S. The lamination convex hull of K, $K^{lc,\Lambda}$, is the smallest lamination convex set containing K. For our purposes, we will call the lamination convex hull simply Λ-convex hull and we will denote it by K^Λ (i.e. here $K^\Lambda := K^{lc,\Lambda}$). However, the original notion of Λ-convexity is defined by duality. We recall the original notion even if it is not the one in use in this thesis: for a compact set K a point does not belong to the Λ-convex hull of K if and only if there exists a Λ-convex function which separates it from K.

In some sense the Λ-convex hull K^Λ constitutes a relaxation of the initial set K. Then one defines *subsolutions* as solutions of the relaxed system, i.e. as solutions z of the linear relations (1.2) which satisfy the relaxed condition $z \in K^\Lambda$. Already at this stage, the concept of subsolutions is reminiscent of the previously introduced concept of short maps for the isometric embedding problem. More precisely, equations (1.1) which define an isometric immersion u can be formulated for the deformation gradient $A := \nabla u$ as the coupling of the linear constraint

$$\operatorname{curl} A = 0$$

with the nonlinear relation

$$A^{\mathrm{T}} A = g.$$

With this interpretation, short maps are "subsolutions" to the isometric embedding problem.

The method of convex integration introduced by Gromov represents a generalization of Nash-Kuiper's result and is based on the upshot that (1.2)-(1.3) admit many interesting solutions if K^Λ is "big enough". Indeed, the key point of convex integration is to reintroduce oscillations by adding suitable localized versions of (1.4) to the subsolutions and to recover a solution of (1.2)-(1.3) iterating this process. The idea of adding oscillatory perturbations can be implemented either in an "implicit way by the so called Baire category method or in a more constructive way. Both approaches provide the key to prove some h-principle-type results for systems of nonlinear evolutionary partial differential equations: they allow to show that, under suitable assumptions on the relaxed set K^Λ, the existence of subsolutions leads to the existence of solutions.

1.3.1.1. *Baire category method.* The Baire category method is a method of enforcing the idea of convex integration and relies on the surprising fact that, in a Baire generic sense, most solutions of the "relaxed system", i.e. most subsolutions, are actually solutions of the original system. Here, we recall the main steps underlying this approach following the "jargon" introduced by Kirchheim in [**Kir03**] (see also [**DLS11**]). In Kirchheim's formalization, the space of subsolutions arises from a nontrivial open set $\mathcal{U} \subset \mathbb{R}^d$ ($\mathcal{U}$ plays the role of K^Λ in the previous section) satisfying the following perturbation property.

Perturbation Property (P): There is a continuous function $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varepsilon(0) = 0$ with the following property: for every $z \in \mathcal{U}$ there is a sequence of solutions $z_j \in C_c^\infty(B_1)$ of (1.2) such that

- $z_j \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^n)$;
- $z + z_j(x) \in \mathcal{U}$ $\forall x \in \mathbb{R}^n$;
- $\int |z_j(x)|^2 \, dx \geq \varepsilon(dist(z, K))$.

The set $\mathcal{U}$ should be thought of as a relaxation of the initial set K, which, according to Kirchheim's jargon, "is stable only near K". Next, define X_0 as follows

$$X_0 := \{z \in C_c^\infty(\Omega) : \ z \text{ satisfies (1.2) and } z(x) \in \mathcal{U} \text{ for all } x \in \Omega\},$$

so that X_0 is the set of smooth compactly supported subsolutions of (1.2)-(1.3). Thanks to the *perturbation property*, X_0 consists of functions which are *perturbable* in an open subdomain $O \subset \Omega$. Then let X be the closure of X_0 with respect to the weak L^2-topology. Assuming that K is bounded, the set X_0 is bounded in L^2 and the topology of weak L^2 convergence is metrizable on X, making it into a complete space. Denote its metric by $d_X(\cdot, \cdot)$. An easy covering argument, together with property (P), results in the following lemma:

LEMMA 1.3.1. *There is a continuous function $\widetilde{\varepsilon} : \mathbb{R}^+ \to \mathbb{R}$ with $\widetilde{\varepsilon}(0) = 0$ such that for every $z \in X_0$ there is a sequence $z_j \in X_0$ with $z_j \overset{d_X}{\to} z$ in X and*

$$\int_\Omega |z_j - z|^2 \, dx \geq \widetilde{\varepsilon}\left(\int_\Omega dist(z(x), K) dx\right).$$

Since the map $I : z \mapsto \int_\Omega |z|^2 \, dx$ is a Baire 1-function on X, an easy application of the Baire category theorem gives that the set

$$Y := \{z \in X : \quad I \text{ is continuous at } z\}$$

is residual in X. By virtue of the previous lemma we can prove that $z \in Y$ implies $z(x) \in K$ for almost every $x \in \Omega$:

THEOREM 1.3.2. *Assuming the perturbation property to hold, the set*

$$Z := \{z \in X : \ z(x) \in K \ a.e. \ x \in \Omega\}$$

is residual in X.

Proof. In order to prove Theorem 1.3.2 it suffices to show that $Y \subset Z$. We will proceed by contradiction. So, let $z \in Y$ and let $z_j \in X_0$ such that $z_j \overset{d_X}{\to} z$ in X. Now, let us assume by contradiction that $\int_\Omega \mathrm{dist}(z(x), K)dx =: \delta > 0$. Thanks to lemma 1.3.1 we can pass (up to a diagonal argument) to a new sequence $\widetilde{z}_j$ with $\widetilde{z}_j \overset{d_X}{\to} z$ and such that

$$\int_\Omega |z_j - \widetilde{z}_j|^2 \, dx \geq \widetilde{\varepsilon}\left(\int_\Omega dist(z_j(x), K)dx\right). \tag{1.6}$$

Since z is a point of continuity of I, it follows that $z_j \to z$ strongly in L^2 as well as $\widetilde{z}_j \to z$ strongly in L^2. This implies in particular that

$$\int_\Omega |z_j - \widetilde{z}_j|^2 \, dx \to 0 \quad \text{as } j \to +\infty. \tag{1.7}$$

Moreover the strong convergence in L^2 of the sequence z_j to z together with the hypothesis of absurd, allow us assume to that, from a certain $\overline{j}$ on, $\int_\Omega \mathrm{dist}(z_j(x), K)dx > \delta/2 > 0$ whence $\widetilde{\varepsilon}\left(\int_\Omega \mathrm{dist}(z_j(x), K)dx\right) > \alpha > 0$ for every $j > \overline{j}$ and for some α. This inequality together with (1.7) contradicts (1.6).

□

1.3.1.2. *Constructive convex integration.* In the previous section we presented the so called Baire category method, which is in some sense non constructive. However, the same idea of adding oscillatory perturbations can be implemented in a constructive way as well. In a nutshell the idea is to define a sequence of subsolutions $z_k \in K^\Lambda$ recursively as

$$z_{k+1}(x) \; = \; z_k(x) + Z_k(x, \lambda_k x), \tag{1.8}$$

where

$$Z_k(x, \xi)$$

is a periodic plane-wave (see (1.4)) solution of (1.2) in the variable ξ, parametrized by x and λ_k is a large frequency to be chosen. The aim is to choose the plane-wave Z_k and the frequency λ_k iteratively in such a way that

- z_k continues to satisfy (1.2) (strictly speaking this requires an additional corrector term in the scheme (1.8));
- z_k belongs to the relaxed constitutive set K^Λ;
- $z_k \to z$ in $L^2(\Omega)$ with $z \in K$ a.e..

The convergence of this constructive scheme is improved by choosing the frequencies λ_k higher and higher. On the other hand clearly any (fractional) derivative or Hölder norm of z_k gets worse by such a choice of λ_k. The best regularity corresponds to the slowest rate at which the frequencies λ_k tend to infinity while still leading to convergence.

1.3.2. Non–standard solutions in fluid dynamics.

1.3.2.1. *Incompressible Euler equations: non-uniqueness results.* The first and leading example of the h-principle in the realm of fluid dynamics is due to De Lellis and Székelyhidi and pertains to the incompressible Euler equations:

$$\begin{cases} \mathrm{div}_x v \ = \ 0 \\ \partial_t v + \mathrm{div}_x (v \otimes v) + \nabla_x p \ = \ 0 \\ v(\cdot, 0) \ = \ v^0 \end{cases} . \tag{1.9}$$

Here the unknowns v and p are, respectively, a vector field and a scalar field defined on $\mathbb{R}^n \times [0, T)$. These fundamental equations were derived 250 years ago by Euler and since then have played a major role in fluid dynamics. There are several outstanding problems connected to (1.9). In particular, weak solutions are known to be badly behaved in the sense of Hadamard's well-posedness: in the groundbreaking paper [**Sch93**] proved the existence of a nontrivial weak solution compactly supported in time. Thanks to the intuition of De Lellis and Székelyhidi such a nonuniqueness result has been explained as a suitable variant of the original h-principle by use of the method of convex integration. Moreover, such an approach allowed them to go way beyond the result of Scheffer and it has lead to new developments for several equations in fluid dynamics included the one presented in this work.

As already mentioned, the first nonuniqueness result for weak solutions of (1.9) is due to Scheffer in [**Sch93**]. The main theorem of [**Sch93**] states the existence of a nontrivial weak solution in $L^2(\mathbb{R}^2 \times \mathbb{R})$ with compact support in space and time. Later on Shnirelman in [**Shn97**] gave a different proof of the existence of a nontrivial weak solution in space-periodic setting and with compact support in time. In these constructions it is not clear whether the solution belongs to the energy space. In the paper [**DLS09**], De Lellis and Székelyhidi provided a relatively simple proof of the following stronger statement.

THEOREM 1.3.3 (Non-uniqueness of weak solutions to the incompressible Euler equations). *There exist infinitely many compactly supported weak solutions of the incompressible Euler equations* (1.9) *in any space dimension greater or equal to 2. In particular there are infinitely many solutions $v \in L^\infty \cap L^2$ to* (1.9) *for $v^0 = 0$ and arbitrary $n \geq 2$.*

The proof of Theorem 1.3.3 is based on the notion of subsolution. The spirit behind the notion of subsolutions in this context is the same as the one outlined in the previous sections for general evolutionary partial differential equations. On the other hand, the definition of subsolution for the incompressible Euler system (1.9) can be made explicit and can be motivated in terms of the Reynold stress (see [**DLS11**] for more details on the connection between Reynold stress and subsolutions). In particular, if one writes (1.9) as the coupling of a linear system of PDEs and a pointwise non-linear constraint (as in (1.2)-(1.3)), then subsolutions are solutions of this linear system which belong pointwise to the convex hull of the non-linear constraint set. In other words:

DEFINITION 1.3.4 (Subsolution of incompressible Euler). *Let $\overline{e} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equations with given kinetic energy density $\overline{e}$ is a triple*

$$(v,u,q) \, : \, \mathbb{R}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}$$

with the following properties:

- $v \in L^2_{loc}$, $u \in L^1_{loc}$, *q is a distribution;*
-
$$\left\{ \begin{array}{rcl} \operatorname{div}_x v &=& 0 \\ \partial_t v + \operatorname{div}_x u + \nabla_x q &=& 0 \end{array} \right. \tag{1.10}$$
in the sense of distributions;
-
$$v \otimes v - u \leq \frac{2}{n}\overline{e} Id \ \ a.e.$$

Observe that subsolutions automatically satisfy $\frac{1}{2}|v|^2 \leq \overline{e}$ a.e. If in addition, the equality sign $\frac{1}{2}|v|^2 = \overline{e}$ a.e. holds true, then the v component of the subsolution is in fact a weak solution of the incompressible Euler equations. From the previous definition we can grasp even better the analogy between the velocity field of the Euler equations and the differential of the embedding in the isometric embedding problem and hence between subsolutions and short maps. Also the heuristic behind the two results shows striking similarities. The key point in De Lellis and Székelyhidi's approach to prove Theorem 1.3.3 is that, starting from a subsolution, an appropriate iteration process reintroduce the high frequencies oscillations. In the limit of this process one obtains weak solutions to (1.9). However, since the oscillations are reintroduced in a very non-unique way, in fact this generates *several* solutions from the same subsolution. The relevant iteration scheme has been already outlined in the general setting in Section 1.3.1. The following theorem comes from Proposition 2 in [**DLS10**] and is a precise formulation of the previous discussion.

THEOREM 1.3.5 (Subsolution criterion). *Let* $\overline{e} \in C(\mathbb{R}^n \times (0,T))$ *and* $(\overline{v}, \overline{u}, \overline{q})$ *a smooth, strict subsolution, i.e.*

$$(\overline{v}, \overline{u}, \overline{q}) \in C^\infty(\mathbb{R}^n \times (0,T)) \textit{ satisfies } (1.10)$$

and

$$v \otimes v - u < \frac{2}{n}\overline{e} Id \textit{ a.e. on } \mathbb{R}^n \times (0,T). \tag{1.11}$$

Then there exist infinitely many weak solutions $v \in L^\infty_{loc}(\mathbb{R}^n \times (0,T))$ *of the incompressible Euler equations* (1.9) *with pressure* $p = \overline{q} - \frac{2}{n}\overline{e}$ *and such that*

$$\frac{1}{2}|v|^2 = \overline{e}$$

for a.e. (x,t). *Infinitely many among these belong to* $C((0,T), L^2)$. *If in addition*

$$\overline{v}(\cdot, t) \rightharpoonup v^0(\cdot) \textit{ in } L^2_{loc}(\mathbb{R}^n) \textit{ as } t \to 0,$$

then all the v*'s so constructed solve* (1.9).

The previous results show that weak solutions of the Euler equations are in general highly non-unique. Moreover the kinetic energy density $\frac{1}{2}|v|^2$ can be prescribed as an independent quantity. Since classical C^1 solutions of the incompressible Euler equations are characterized by conservation of the total kinetic energy

$$\frac{d}{dt}\int_{\mathbb{R}^n} \frac{|v|^2}{2}(x,t)dx = 0,$$

one can complement the notion of weak solution to (1.9) with several admissibility criteria defined as "relaxations" (in a proper sense) of the energy conservation. Let us denote by $L^2_w(\mathbb{R}^n)$ the space $L^2(\mathbb{R}^n)$ endowed with the weak topology. We recall that any weak solution of (1.9) can be modified on a set of measure zero so to get $v \in C([0,T), L^2_w(\mathbb{R}^n))$. Consequently v has a well-defined trace at every time. This allows to introduce the following admissibility criteria for weak solutions:

(a)
$$\int |v|^2 (x,t)dx \le \int |v_0|^2 (x)dx \qquad \text{for every } t.$$

(b)
$$\int |v|^2 (x,t)dx \le \int |v|^2 (x,s)dx \qquad \text{for every } t > s.$$

(c) If in addition $v \in L^3_{loc}$, then
$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \le 0$$
in the sense of distributions.

The first two criteria are of course suggested by the conservation of kinetic energy of classical solutions, while condition (c) has been proposed for the first time by Duchon and Robert in [**DR00**] and it resembles the admissibility criteria which are popular in the literature on hyperbolic conservation laws. However, none of these criteria restore the uniqueness of weak solutions.

Theorem 1.3.6 (Non-uniqueness of admissible weak solutions to the incompressible Euler equations). *There exist initial data $v^0 \in L^\infty \cap L^2$ for which there are infinitely many bounded solutions of* (1.9) *which are strongly L^2-continuous and satisfy (a), (b) and (c).*

The conditions (a), (b) and (c) hold with the equality sign for infinitely many of these solutions, whereas for infinitely many other they hold as strict inequalities.

This theorem has been stated and proved in [**DLS10**]. The second part of the statement generalizes the intricate construction of Shnirelman in [**Shn97**], which produced the first example of a weak solution in 3d of (1.9) with strict inequalities in (a) and (b).

1.3.2.2. *Incompressible Euler equations: h-principle.* De Lellis and Székelyhidi in [**DLS12**] were able to extend the previous results: they devised a new iteration scheme which produces continuous and even Hölder continuous solutions on $\mathbb{T}^3$. Indeed, for the incompressible Euler equations, the natural space for convex integration is C^0. The method used in [**DLS09**] producing solutions in L^∞ was a weak form of convex integration. The new iteration scheme of [**DLS12**] is closer to the approach of Nash [**Nas54**] for the isometric embedding problem.

Recently, A. Choffrut [**Cho12**] established optimal h-principles in two and three space dimensions. More specifically he identifies all subsolutions (defined in a suitable sense) which can be approximated in the H^1-norm by exact solutions of the incompressible Euler equations. For the precise statement we refer the reader directly to [**Cho12**].

1.3.2.3. *Incompressible Euler equations: Wild initial data.* The initial data v^0 constructed in Theorem 1.3.6 are obviously not regular, since for regular initial data the local existence theorems and the weak-strong uniqueness (see [**BDLS11**]) ensure local uniqueness under condition (a). One might therefore ask how large is the set of these "wild" initial data. A consequence of De Lellis and Székelyhidi's method is the following density theorem proved by Székelyhidi and Wiedemann in [**SW11**].

THEOREM 1.3.7 (Density of wild initial data). *The set of initial data v^0 for which the conclusions of Theorem 1.3.6 holds is dense in the space of L^2 solenoidal vectorfields.*

Another surprising corollary is that the usual shear flow is a "wild initial datum". More precisely, if we consider the following solenoidal vector field in $\mathbb{R}^2$

$$v^0(x) := \begin{cases} (1,0) & \text{if } x_2 > 0, \\ (-1,0) & \text{if } x_2 < 0, \end{cases} \tag{1.12}$$

then:

THEOREM 1.3.8 (Wild vortex sheet). *For v^0 as in* (1.12)*, there are infinitely many weak solutions of* (1.9) *on* $\mathbb{R}^2 \times [0,\infty)$ *which satisfy (c).*

This theorem has been proven in [**Sz11**] using Proposition 1.3.5 and hence the proof amounts to showing the existence of a suitable subsolution. We will further discuss this result in Chapter 3.

1.3.2.4. *Incompressible Euler equations: global existence of weak solutions.* A further application of Theorem (1.3.5) is due to Wiedemann [**Wie11**]. E. Wiedemann considered an arbitrary initial datum $v^0 \in L^2_{loc}(\mathbb{R}^n)$ and constructed a smooth triple $(\overline{v}, \overline{u}, \overline{q}) \in C^\infty(\mathbb{R}^n \times (0,T))$ which solves (1.10) with initial datum v^0 and is a subsolution for a proper choice of the profile of $\overline{e}$. In particular, by constructing a subsolution with bounded energy, Wiedemann in [**Wie11**] recently obtained the following:

COROLLARY 1.3.9 (Global existence for weak solutions). *Let $v^0 \in L^2(\mathbb{T}^n)$ be a solenoidal vector field. Then there exist infinitely many global weak solutions of* (1.9) *with bounded energy.*

1.3.2.5. *Active scalar equations.* Active scalar equations are a class of systems of evolutionary partial differential equations in n space dimensions. The unknowns are the "active" scalar function θ and the velocity field v, which, for simplicity, is a divergence-free vector field. The equations are

$$\begin{cases} \mathrm{div}_x v \;=\; 0 \\ \partial_t \theta + v \cdot \nabla_x \theta \;=\; 0 \end{cases} \tag{1.13}$$

and v and θ are coupled by an integral operator, namely

$$v \;=\; T[\theta]. \tag{1.14}$$

Several systems of partial differential equations in fluid dynamics fall into this class. One can rewrite (1.13)-(1.14) in the spirit of Section 1.3.1, as a system of linear relations

$$\begin{cases} \mathrm{div}_x v \;=\; 0 \\ \partial_t \theta + \mathrm{div}_x\, q \;=\; 0 \\ v \;=\; T[\theta] \end{cases} \tag{1.15}$$

coupled with the nonlinear constraint

$$q \;=\; \theta v. \tag{1.16}$$

The initial value problem for the system (1.15)-(1.16) amounts to prescribing $\theta(x,0) \;=\; \theta_0(x)$. As described in Section 1.3.1 a key point is that the linear relations (1.15) admit a large set of plane wave solutions. Note that these linear relations are not strictly speaking of the form (1.2) and in order to define a suitable analogue of the plane waves in this setting, the linear operator T can be assumed to be translation invariant. Let $m(\xi)$ be its corresponding Fourier multiplier. Then one requires in addition that $m(\xi)$ is $0-$homogeneous so that (1.15) has the same scaling invariance as (1.2). Furthermore, the constraint $\operatorname{div}_x v = 0$ implies that $\xi \cdot m(\xi) = 0$. In spite of this restriction, several interesting equations fall into this category. Perhaps the best known examples are the surface quasi geostrophic and the incompressible porous medium equations, corresponding respectively to

$$m(\xi) = i\,|\xi|^{-1}\,(-\xi_2, \xi_1) \quad \text{and} \tag{1.17}$$

$$m(\xi) = |\xi|^{-2}\,(\xi_1\xi_2, -\xi_1^2). \tag{1.18}$$

In [**CFG09**] Cordoba, Faraco and Gancedo proved the following theorem.

Theorem 1.3.10. *Assume m is given by* (1.18)*. Then there exist infinitely many weak solutions of* (1.15) *and* (1.16) *in* $L^\infty(\mathbb{T}^2 \times [0, +\infty[)$ *with* $\theta_0 = 0$.

This was generalized by Shvydkoy to all even $m(\xi)$ satisfying a mild additional regularity assumption. We refer to the original paper [**Shv11**] for the details.

The proof of Theorem 1.3.10 in [**CFG09**], as well as the proof by Shvydkoy in [**Shv11**], relies on some refined tools which were developed in the theory of laminates and differential inclusions and they present some substantial differences with the methods of De Lellis and Székelyhidi in [**DLS09**] and [**DLS10**]. Indeed, the method used in [**CFG09**] is still based on understanding the equation as a differential inclusion in the spirit of Tartar [**Tar79**], but in the context of the porous media equation the situation differs from the setting of [**DLS09**]-[**DLS10**] and the authors had to take different routes in several steps. First, we would like to recall that a central point is to find an open set $\mathcal{U}$ satisfying the perturbation property (P). One possible candidate would be to take the largest open set $\mathcal{U}_{max}$ satisfying (P). Obviously this set is particularly meaningful since it gives the largest possible space X for which genericity conclusions holds. Moreover, this has the advantage that - at least in many relevant cases - the set $\mathcal{U}_{max}$ coincides with the interior of the Λ–convex hull K^Λ, which in turn can be characterized by separation arguments. For instance, in Theorem

1.3.5 condition (1.11) characterizes precisely the interior of K^Λ. Furthermore, in this case the interior of K^Λ is the interior of the convex hull K^{co}.

In [**CFG09**] and [**Shv11**] the authors avoid calculating the full Λ–convex hull and instead restrict themselves to exhibiting a nontrivial (but possibly much smaller) open set $\mathcal{U}$ satisfying (P). Opposite to the incompressible Euler equations, in [**CFG09**] and [**Shv11**] the Λ–convex hull does not agree with the convex hull and more relevant $K \subset \partial K^\Lambda$. This is of course an obstruction for the available versions of convex integration as presented in Section 1.3.1 (the ones based on Baire category and the direct constructions). So the argument in [**CFG09**] and [**Shv11**] suggests a more systematic approach: instead of fixing a set and computing the hull, they pick a reasonable matrix A and compute $(A+\Lambda)\cap K$. Then by the results in [**Kir03**] it is enough to find a set $\widetilde{K} \subset (A+\Lambda)\cap K$ such that $A \in \widetilde{K}^{co}$ to find what are called degenerate T_4 configurations (see [**KMS03**]) supported in $\widetilde{K}^{co}$). However, in exchange they are forced to use much more complicated sequences z_j. Indeed, the z_j's used in [**DLS10**] are localizations of simple plane waves, whereas the ones used in [**CFG09**] and [**Shv11**] arise as an infinite nested sequence of repeated plane waves.

The obvious advantage of the method introduced in [**CFG09**] and used in [**Shv11**] is that it seems to be fairly robust and general. It is useful in cases where an explicit computation of the hull K^Λ is out of reach due to the high complexity and high dimensionality. Anyway, the constructions carried out in this dissertation are analogue to the ones by De Lellis and Székelyhidi and they do not require any understanding of the ideas by [**CFG09**] and [**Shv11**].

CHAPTER 2

Non–uniqueness of entropy solutions with arbitrary density

In this chapter we present and prove the first result stated in the Introduction of the thesis, i.e. Theorem 0.2.1: given any continuously differentiable initial density, we can construct bounded initial velocities for which admissible solutions to the isentropic compressible Euler equations are not unique in more than one space dimension.

The content of this chapter corresponds to the subject of the paper [**Chi11**] written by the author during the PhD studies. In particular the structure of the chapter is as follows. In the first section, we introduce the problem and the setting we will work with and we state the main result: even if the equations under investigation have already been presented in the Introduction of the thesis, we chose to recall them herein so that the chapter will be self-contained. Section 2.2 is an overview on the definitions of weak and admissible solutions and gives a first glimpse on how Theorem 0.2.1 is achieved. Section 2.3 is devoted to the reformulation of a simplified version of the isentropic compressible Euler equations as a differential inclusion and to the corresponding geometrical analysis. In Section 2.4 we state and prove a criterion (Proposition 2.4.1) to select initial momenta allowing for infinitely many solutions. The proof builds upon a refined version of the Baire category method for differential inclusions developed in [**DLS10**] and aimed at yielding weakly continuous in time solutions. Section 2.5 and 2.6 contain the proofs of the main tools used to prove Proposition 2.4.1. In Section 2.7, we show initial momenta satisfying the requirements of Proposition 2.4.1. Finally, in Section 2.8 we prove Theorem 0.2.1 (here stated in the first section as Theorem 2.1.1) by applying Proposition 2.4.1.

2.1. Introduction

We deal with the Cauchy Problem for the isentropic compressible Euler equations in the space-periodic setting.

We first introduce the isentropic compressible Euler equations of gas dynamics in n space dimensions, $n \geq 2$ (cf. Section 3.3 of [**Daf10**]). They are obtained as a simplification of the full compressible Euler equations, by assuming the entropy to be constant. The state of the gas will be described through the state vector

$$V = (\rho, m)$$

whose components are the density ρ and the linear momentum m. In contrast with the formulation of the problem given in the Introduction of the thesis, here the equations are written in terms of the linear momentum field which allows to write the equations in the *canonical* form (see [**Daf10**]). The balance laws in force are for mass and linear momentum. The resulting system, which consists of $n+1$ equations, takes the form (cf. (0.1)):

$$\begin{cases} \partial_t \rho + \mathrm{div}_x m \ = \ 0 \\ \partial_t m + \mathrm{div}_x \left(\frac{m \otimes m}{\rho} \right) + \nabla_x [p(\rho)] \ = \ 0 \\ \rho(\cdot, 0) \ = \ \rho^0 \\ m(\cdot, 0) \ = \ m^0 \end{cases} . \tag{2.1}$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas in question. A common choice is the polytropic pressure law

$$p(\rho) = k\rho^\gamma$$

with constants $k > 0$ and $\gamma > 1$. The set of admissible values is $P = \{\rho > 0\}$ (cf. [**Daf10**] and [**Ser99**]). As already explained, the system is hyperbolic if

$$p'(\rho) > 0.$$

In addition, thermodynamically admissible processes must also satisfy an additional constraint coming from the energy inequality

$$\partial_t \left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho} \right) + \mathrm{div}_x \left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \right] \ \leq \ 0 \tag{2.2}$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$. The physical region for (2.1) is $\{(\rho, m) | \, |m| \leq R\rho\}$, for some constant $R > 0$. For $\rho > 0$, $v = m/\rho$ represents the velocity of the fluid.

We will consider, from now on, the case of general pressure laws given by a function p on $[0, \infty[$, that we always assume to be continuously differentiable on $[0, \infty[$. The crucial requirement we impose upon p is that it has to be strictly increasing on $[0, \infty[$. Such a condition is meaningful from a physical viewpoint since it is a consequence of the principles of thermodynamics.

Using some techniques introduced by De Lellis-Székelyhidi (cf. [**DLS09**] and [**DLS10**]) we can consider any continuously differentiable periodic initial density ρ^0 and exhibit suitable periodic initial momenta m^0 for which space-periodic weak admissible solutions of (2.1) are not unique on some finite time-interval.

THEOREM 2.1.1. *Let $n \geq 2$. Then, for any given function p and any given continuously differentiable periodic initial density ρ^0, there exist a bounded periodic initial momentum m^0 and a positive time $\overline{T}$ for which there are infinitely many space-periodic admissible solutions (ρ, m) of* (2.1) *on $\mathbb{R}^n \times [0, \overline{T}[$ with $\rho \in C^1(\mathbb{R}^n \times [0, \overline{T}[)$.*

REMARK 2.1.2. *Indeed, in order to prove Theorem 2.1.1, it would be enough to assume that the initial density is a Hölder continuous periodic function: $\rho^0 \in C^{0,\alpha}(\mathbb{R}^n)$ (cf. Proof of Proposition 2.7.1).*

Some connected results are obtained in [**DLS10**] (cf. Theorem 2 therein) as a further consequence of their analysis on the incompressible Euler equations. Inspired by their approach, we adapt and apply directly to (2.1) the method of convex integration combined with Tartar's programme on oscillation phenomena in conservation laws (see [**Tar79**] and [**KMS03**]). In this way, we can show failure of uniqueness of admissible solutions to the compressible Euler equations starting from any given continuously differentiable initial density.

2.2. Weak and admissible solutions to the isentropic Euler system

The deceivingly simple-looking system of first-order partial differential equations (2.1) has a long history of important contributions over more than two centuries. We recall a few classical facts on this system (see for instance [**Daf10**] and [**Lio96**] for more details).

- If ρ^0 and m^0 are "smooth" enough (see Theorem 5.3.1 in [**Daf10**]), there exists a maximal time interval $[0,T[$ on which there exists a unique "smooth" solution (ρ, m) of (2.1) (for $0 \leq t < T$). In addition, if $T < \infty$, and this is the case in general, (ρ, m) becomes discontinuous as t goes to T.
- If we allow for discontinuous solutions, i.e., for instance, solutions $(\rho, m) \in L^\infty$ satisfying (2.1) in the sense of distributions, then solutions are neither unique nor stable. More precisely, one can exhibit sequences of such solutions which converge weakly in $L^\infty - *$ to functions which do not satisfy (2.1).
- In order to restore the stability of solutions and (possibly) the uniqueness, one may and should impose further restrictions on bounded solutions of (2.1), restrictions which are known as (Lax) entropy inequalities.

In this chapter we address the problem of better understanding the efficiency of entropy inequalities as selection criteria among weak solutions.

Here, we have chosen to emphasize the case of the flow with *space periodic* boundary conditions. For *space periodic* flows we assume that the fluid fills the entire space $\mathbb{R}^n$ but with the condition that m, ρ are periodic functions of the space variable. The space periodic case is not a physically achievable one, but it is relevant on the physical side as a model for some flows. On the mathematical side, it retains the complexities due to the nonlinear terms (introduced by the kinematics) and therefore it includes many of the difficulties encountered in the general case. However the former is simpler to treat because of the absence of boundaries. Furthermore, using Fourier transform as a tool simplifies the analysis.

Let $Q = [0,1]^n$, $n \geq 2$ be the unit cube in $\mathbb{R}^n$. We denote by $H^m_p(Q)$, $m \in \mathbb{N}$, the space of functions which are in $H^m_{loc}(\mathbb{R}^n)$ and which are periodic with period Q:

$$f(x+l) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and every } l \in \mathbb{Z}^n.$$

For $m = 0$, $H^0_p(Q)$ coincides simply with $L^2(Q)$. Analogously, for every functional space X we define $X_p(Q)$ to be the space of functions which are locally (over $\mathbb{R}^n$) in X and are periodic of period Q. The functions in $H_p^m(Q)$ are easily characterized by their Fourier series expansion

(2.3)
$$H_p^m(Q) = \left\{ f \in L_p^2(Q) : \sum_{k\in\mathbb{Z}^n} |k|^{2m} \left|\widehat{f}(k)\right|^2 < \infty \text{ and } \widehat{f}(0) = 0 \right\},$$

where $\widehat{f} : \mathbb{Z}^n \to \mathbb{C}^n$ denotes the Fourier transform of f. We will use the notation $H(Q)$ for $H_p^0(Q)$ and $H_w(Q)$ for the space $H(Q)$ endowed with the weak L^2 topology.

Let T be a fixed positive time. By a *weak solution* of (2.1) on $\mathbb{R}^n \times [0,T[$ we mean a pair $(\rho, m) \in L^\infty([0,T[; L_p^\infty(Q))$ satisfying

(2.4)
$$|m(x,t)| \leq R\rho(x,t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,T[\text{ and some } R > 0,$$

and such that the following identities hold for every test functions $\psi \in C_c^\infty([0,T[; C_p^\infty(Q))$, $\phi \in C_c^\infty([0,T[; C_p^\infty(Q))$:

$$\int_0^T \int_Q [\rho\partial_t\psi + m \cdot \nabla_x\psi]\, dxdt + \int_Q \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{2.5}$$

$$\int_0^T \int_Q \left[m \cdot \partial_t\phi + \left\langle \frac{m\otimes m}{\rho}, \nabla_x\phi \right\rangle + p(\rho)\operatorname{div}_x \phi \right] dxdt$$
$$+ \int_Q m^0(x) \cdot \phi(x,0)dx \;=\; 0. \tag{2.6}$$

For $n \geq 2$ the only non-trivial entropy is the total energy $\eta = \rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}$ which corresponds to the flux $\Psi = \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right)m$.

Then a bounded weak solution (ρ, m) of (2.1) satisfying (2.2) in the sense of distributions, i.e. satisfying the following inequality

(2.7)
$$\int_0^T \int_Q \left[\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}\right)\partial_t\varphi + \left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right) m \cdot \nabla_x\varphi \right]$$
$$+ \int_Q \left(\rho^0\varepsilon(\rho^0) + \frac{1}{2}\frac{|m^0|^2}{\rho}\right)\varphi(\cdot,0) \;\geq\; 0,$$

for every nonnegative $\varphi \in C_c^\infty([0,T[; C_p^\infty(Q))$, is said to be an *entropy* (or *admissible*) solution of (2.1).

The lack of entropies is one of the essential reasons for a very limited understanding of compressible Euler equations in dimensions greater than or equal to 2.

The recent paper [**DLS10**] by De Lellis-Székelyhidi gives an example in favour of the conjecture that entropy solutions to the multidimensional compressible Euler equations are in general not unique: see Theorem 0.1.1 in the Introduction of the thesis. Showing that this conjecture is true has far-reaching consequences (see also [**Ell06**]). The entropy condition is not sufficient as a selection principle for physical/unique solutions. The non-uniqueness result by De Lellis-Székelyhidi is a byproduct of their new analysis of the incompressible Euler equations based on its formulation as a differential inclusion. They first show that, for some bounded compactly supported initial data, none of the classical admissibility criteria singles out a unique solution to the Cauchy problem for the incompressible Euler equations. As a consequence, by constructing a piecewise constant in space and independent of time density ρ, they look at the compressible isentropic system as a "piecewise incompressible" system (i.e. still incompressible in the support of the velocity field) and thereby exploit the result for the incompressible Euler equations to exhibit bounded initial density and bounded compactly supported initial momenta for which admissible solutions of (2.1) are not unique (in more than one space dimension).

Inspired by their techniques, we give a further counterexample to the well-posedeness of entropy solutions to (2.1). Our result differs in two main aspects: here the initial density can be any given "regular" function and remains "regular" forward in time while in [**DLS10**] the density allowing for infinitely many admissible solutions must be chosen as piecewise constant in space; on the other hand we are not able to deal with compactly supported momenta (indeed we work in the periodic setting), hence our non-unique entropy solutions are only locally L^2 in contrast with the global-L^2-in-space property of solutions obtained in [**DLS10**]. Moreover, we have chosen to study the case of the flow in a cube of $\mathbb{R}^n$ with *space periodic* boundary conditions. This case leads to many technical simplifications while retaining the main structure of the problem.

More precisely, we are able to analyze the compressible Euler equations in the framework of convex integration introduced in Chapter 1. We recall that this method works well with systems of nonlinear PDEs such that the convex envelope (in an appropriate sense) of each small domain of the submanifold representing the PDE in the jet-space (see [**EM02**] for more details) is big enough. In our case, we consider a simplification of system (2.1), namely the semi-stationary associated problem, whose submanifold allows a convex integration approach leading us to recover the result of Theorem 2.1.1.

We are interested in the semi-stationary Cauchy problem associated with the isentropic Euler equations (simply set to 0 the time derivative of the density in (2.1) and drop the initial condition for ρ):

$$\begin{cases} \mathrm{div}_x m \;=\; 0 \\ \partial_t m + \mathrm{div}_x\left(\frac{m\otimes m}{\rho}\right) + \nabla_x[p(\rho)] \;=\; 0 \\ m(\cdot,0) \;=\; m^0. \end{cases} \tag{2.8}$$

A pair $(\rho, m) \in L^\infty_p(Q)\times L^\infty([0,T[; L^\infty_p(Q))$ is a *weak solution* on $\mathbb{R}^n \times [0,T[$ of (2.8) if $m(\cdot,t)$ is weakly-divergence free for almost every $0 < t < T$ and satisfies the following bound

$$|m(x,t)| \le R\rho(x) \qquad \text{for a.e. } (x,t)\in\mathbb{R}^n\times[0,T[\text{ and some } R>0, \tag{2.9}$$

and if the following identity holds for every $\phi \in C^\infty_c([0,T[; C^\infty_p(Q))$:

$$\int_0^T\int_Q \left[m\cdot\partial_t\phi + \left\langle \frac{m\otimes m}{\rho}, \nabla_x\phi\right\rangle + p(\rho)\,\mathrm{div}_x\,\phi\right]dxdt + \int_Q m^0(x)\cdot\phi(x,0)dx \;=\; 0. \tag{2.10}$$

A general observation suggests us that a non-uniqueness result for weak solutions of (2.8) whose momentum's magnitude satisfies some suitable constraint could lead us to a non-uniqueness result for entropy solutions of the isentropic Euler equations (2.1). Indeed, the entropy solutions we construct in Theorem 2.1.1 come from *some* weak solutions of (2.8).

THEOREM 2.2.1. *Let $n \ge 2$. Then, for any given function p, any given density $\rho_0 \in C^1_p(Q)$ and any given finite positive time T, there exists a bounded initial momentum m^0 for which there are infinitely many weak solutions $(\rho, m) \in C^1_p(Q) \times C([0,T]; H_w(Q))$ of (2.8) on $\mathbb{R}^n \times [0,T[$ with density $\rho(x) = \rho_0(x)$.*
In particular, the obtained weak solutions m satisfy

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad a.e.\ in\ \mathbb{R}^n\times[0,T[, \tag{2.11}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad a.e.\ in\ \mathbb{R}^n, \tag{2.12}$$

for some smooth function χ.

An easy computation shows how, by properly choosing the function χ in (2.11)-(2.12), the solutions (ρ_0, m) of (2.8) obtained in Theorem 2.2.1 satisfy the admissibility condition (2.7).

THEOREM 2.2.2. *Under the same assumptions of Theorem 2.2.1, there exists a maximal time* $\overline{T} > 0$ *such that the weak solutions* (ρ, m) *of* (2.8) *(coming from Theorem 2.2.1) satisfy the admissibility condition* (2.7) *on* $[0, \overline{T}[$.

Our construction yields initial data m^0 for which the nonuniqueness result of Theorem 2.1.1 holds on any time interval $[0, T[$, with $T \leq \overline{T}$. However, as pointed out before, for sufficiently regular initial data, classical results give the local uniqueness of smooth solutions. Thus, *a fortiori*, the initial momenta considered in our examples have necessarily a certain degree of irregularity.

2.3. Geometrical analysis

This section is devoted to a qualitative analysis of the isentropic compressible Euler equations in a semi-stationary regime (i.e. (2.8)).

As in [**DLS09**] we will interpret the system (2.8) in terms of a differential inclusion, so that it can be studied in the framework combining the plane wave analysis of Tartar, the convex integration of Gromov and the Baire's arguments. For a complete description of this general framework we refer to Chapter 1, Section 1.3.1. In Section 2.3.2 we will recall only the tools needed for our construction.

2.3.1. Differential inclusion. The system (2.8) can indeed be naturally expressed as a linear system of partial differential equations coupled with a pointwise nonlinear constraint, usually called *differential inclusion*.

The following Lemma, based on Lemma 2 in [**DLS10**], gives such a reformulation. We will denote by $\mathcal{S}^n$ the space of symmetric $n \times n$ matrices, by $\mathcal{S}^n_0$ the subspace of $\mathcal{S}^n$ of matrices with null trace, and by I_n the $n \times n$ identity matrix.

LEMMA 2.3.1. *Let* $m \in L^\infty([0, T]; L^\infty_p(Q; \mathbb{R}^n))$,
$U \in L^\infty([0, T]; L^\infty_p(Q; \mathcal{S}^n_0))$ *and* $q \in L^\infty([0, T]; L^\infty(Q; \mathbb{R}^+))$ *such that*

$$\begin{aligned} \operatorname{div}_x m &= 0 \\ \partial_t m + \operatorname{div}_x U + \nabla_x q &= 0. \end{aligned} \tag{2.13}$$

If (m, U, q) *solve* (2.13) *and in addition there exists* $\rho \in L^\infty_p(\mathbb{R}^n; \mathbb{R}^+)$ *such that* (5.7) *holds and*

$$U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n \quad a.e.\ in\ \mathbb{R}^n \times [0, T],$$
$$q = p(\rho) + \frac{|m|^2}{n\rho} \quad a.e.\ in\ \mathbb{R}^n \times [0, T], \tag{2.14}$$

then m *and* ρ *solve* (2.8) *distributionally. Conversely, if* m *and* ρ *are weak solutions of* (2.8)*, then* m*,* $U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n$ *and* $q = p(\rho) + \frac{|m|^2}{n\rho}$ *solve* (2.13)*-*(2.14)*.*

In Lemma 2.3.1 we made clear the distinction between the augmented system (2.13), whose linearity allows a plane wave analysis, and the nonlinear pointwise constraint (2.14), which leads us to study the graph below.

For any given $\rho \in]0, \infty[$, we define the following graph

$$K_\rho := \Bigg\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n,$$
$$q = p(\rho) + \frac{|m|^2}{n\rho} \Bigg\}. \tag{2.15}$$

The key of the forthcoming analysis is the behaviour of the graph K_ρ with respect to the wave vectors associated with the linear system (2.13): are differential and algebraic constraints in some sense compatible?

For our purposes, it is convenient to consider "slices" of the graph K_ρ, by considering vectors m whose modulus is subject to some ρ-depending condition. Thus, for any given $\chi \in \mathbb{R}^+$, we define:

$$K_{\rho,\chi} := \Bigg\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : U = \frac{m \otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n,$$
$$q = p(\rho) + \frac{|m|^2}{n\rho}, |m|^2 = \rho\chi \Bigg\}. \tag{2.16}$$

2.3.2. Wave cone. For the sake of completeness, we remind the notions of *planewave* solutions and *wave cone*, previously introduced in

Section 1.3.1. According to Tartar's framework [**Tar79**], we consider a system of first order linear PDEs (see (1.2))

$$\sum_i A_i \partial_i z = 0 \tag{2.17}$$

where z is a vector valued function and the A_i are matrices. Then, *planewave* solutions to (2.17) are solutions of the form

$$z(x) = a h(x \cdot \xi), \tag{2.18}$$

with $h : \mathbb{R} \to \mathbb{R}$. In order to find such solutions, we have to solve the relation $\sum_i \xi_i A_i a = 0$, where ξ_i is the oscillation frequency in the direction i. The set of directions a for which a solution $\xi \neq 0$ exists is called *wave cone* Λ of the system (2.17): equivalently Λ characterizes the directions of one dimensional oscillations compatible with (2.17).

The system (2.13) can be analyzed in this framework. Consider the $(n+1) \times (n+1)$ symmetric matrix in block form

$$M = \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix}. \tag{2.19}$$

Note that, with the new coordinates $y = (x, t) \in \mathbb{R}^{n+1}$, the system (2.13) can be easily rewritten as $\operatorname{div}_y M = 0$ (the divergence of M in space-time is zero). Thus, the wave cone associated with the system (2.13) is equal to

$$\Lambda = \left\{ (m, U, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}^+ : \det \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix} = 0 \right\}. \tag{2.20}$$

Indeed, the relation $\sum_i \xi_i A_i a = 0$ for the system (2.13) reads simply as $M \cdot (\xi, c) = 0$, where $(\xi, c) \in \mathbb{R}^n \times \mathbb{R}$ (ξ is the space-frequency and c the time-frequency): this equation admits a non-trivial solution if M has null determinant, hence (2.20).

2.3.3. Convex hull and geometric setup. Since it will be of great importance in this chapter, we formulate once more the definition of Λ-convex hull already given in Section 1.3.1.

Given a cone Λ, we say that a set S is convex with respect to Λ (or Λ-convex) if, for any two points $A, B \in S$ with $B - A \in \Lambda$, the whole segment $[A, B]$ belongs to S. The Λ-convex hull of $K_{\rho,\chi}$ is the smallest Λ-convex set $K^\Lambda_{\rho,\chi}$ containing $K_{\rho,\chi}$, i.e. the set of states obtained by mixture of states of $K_{\rho,\chi}$ through oscillations in Λ-directions (Gromov [**Gro86**], who works in the more general setting of jet bundles, calls this the P- convex hull). The key point in Gromov's method of convex integration (which is a far reaching generalization of the work of Nash [**Nas54**] and Kuiper [**Kui95**] on isometric immersions) is that (2.17) coupled with a pointwise nonlinear constraint of the form $z \in K$ a.e. admits many interesting solutions provided that the Λ-convex hull of K, K^Λ, is sufficiently large. In applications to elliptic and parabolic systems we always have $K^\Lambda = K$ so that Gromov's approach does not directly apply. For other applications to partial differential equations it turns out that one can work with the Λ-convex hull defined by duality. More precisely, a point does not belong to the Λ- convex hull defined by duality if and only if there exists a Λ-convex function which separates it from K. A crucial fact is that the second notion is much weaker. This surprising fact is illustrated in [**KMS03**].

In our case, the wave cone is quite large, therefore it is sufficient to consider the stronger notion of Λ-convex hull, indeed it coincides with the whole convex hull of $K_{\rho,\chi}$.

Lemma 2.3.2. *For any $S \in \mathcal{S}^n$ let $\lambda_{max}(S)$ denote the largest eigenvalue of S. For $(\rho, m, U) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n_0$ let*

$$e(\rho, m, U) := \lambda_{max}\left(\frac{m \otimes m}{\rho} - U\right). \tag{2.21}$$

Then, for any given $\rho, \chi \in \mathbb{R}^+$, the following holds

(i) *$e(\rho, \cdot, \cdot) : \mathbb{R}^n \times \mathcal{S}^n_0 \to \mathbb{R}$ is convex;*
(ii) *$\frac{|m|^2}{n\rho} \le e(\rho, m, U)$, with equality if and only if $U = \frac{m\otimes m}{\rho} - \frac{|m|^2}{n\rho} I_n$;*
(iii) *$|U|_\infty \le (n-1)e(\rho, m, U)$, with $|U|_\infty$ being the operator norm of the matrix;*
(iv) *the $\frac{\chi}{n}$-sublevel set of e defines the convex hull of $K_{\rho,\chi}$, i.e.*

$$K^{co}_{\rho,\chi} = \left\{(m, U, q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : e(\rho, m, U) \le \frac{\chi}{n},\right.$$
$$\left. q = p(\rho) + \frac{\chi}{n}\right\} \tag{2.22}$$

and $K_{\rho,\chi} = K^{co}_{\rho,\chi} \cap \{|m|^2 = \rho\chi\}$.

For the proof of (i)-(iv) we refer the reader to the proof of Lemma 3.2 in [**DLS10**]: the arguments there can be easily adapted to our case.

We observe that, for any $\rho, \chi \in \mathbb{R}^+$, the convex hull $K^{co}_{\rho,\chi}$ lives in the hyperplane H of $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ defined by $H := \Big\{(m, U, q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : q = p(\rho) + \frac{\chi}{n}\Big\}$. Therefore, the interior of $K^{co}_{\rho,\chi}$ as a subset of $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ is empty. This seems to prevent us from working in the classical framework of convex integration, but we can overcome this apparent obstacle.

For any $\rho, \chi \in \mathbb{R}^+$, we define the *hyperinterior* of $K^{co}_{\rho,\chi}$, and we denote it with "hint $K^{co}_{\rho,\chi}$", as the following set

$$\text{hint } K^{co}_{\rho,\chi} := \Bigg\{(m, U, q) \in \mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+ : e(\rho, m, U) < \frac{\chi}{n},$$

$$q = p(\rho) + \frac{\chi}{n}\Bigg\}. \tag{2.23}$$

In the framework of convex integration, the larger the Λ-convex hull of $K_{\rho,\chi}$ is, the bigger the breathing space will be. How to "quantify" the meaning of a "large" Λ-convex hull in our context? The previous definition provides an answer: the Λ-convex hull of $K_{\rho,\chi}$ will be "large" if its hyperinterior is nonempty. The wave cone of the semi-stationary Euler isentropic system is wide enough to ensure that the Λ-convex hull of $K_{\rho,\chi}$ coincides with the convex hull of $K_{\rho,\chi}$ and has a nonempty hyperinterior. As a consequence, we can construct irregular solutions oscillating along any fixed direction. For our purposes, it will be convenient to restrict to some *special directions* in Λ, consisting of matrices of rank 2, which are not stationary in time, but are associated with a constant pressure.

Lemma 2.3.3. *Let $c, d \in \mathbb{R}^n$ with $|c| = |d|$ and $c \neq d$, and let $\rho \in \mathbb{R}^+$.*
Then $\left(c - d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right) \in \Lambda$.

Proof. Since the vector $\left(c + d, -\left(\frac{|c|^2 + c\cdot d}{\rho}\right)\right)$ is in the kernel of the matrix

$$C = \begin{pmatrix} \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho} & c - d \\ c - d & 0 \end{pmatrix},$$

C has indeed determinant zero, hence $\left(c - d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right) \in \Lambda$. $\square$

Now, we introduce some important tools: they allow us to prove that $K^{\Lambda}_{\rho,\chi} = K^{co}_{\rho,\chi}$ is sufficiently large, thus providing us room to find many solutions for (2.13)-(2.14).

As first, we define the *admissible segments* as segments in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ whose directions belong to the wave cone Λ for the linear system of PDEs (2.13) and are indeed *special directions* in the sense specified by Lemma 2.3.3.

Definition 2.3.4. *Given $\rho, \chi \in \mathbb{R}^+$ we call σ an admissible segment for (ρ, χ) if σ is a line segment in $\mathbb{R}^n \times \mathcal{S}^n_0 \times \mathbb{R}^+$ satisfying the following conditions:*

- *σ is contained in the hyperinterior of $K^{co}_{\rho,\chi}$;*
- *σ is parallel to $\left(c-d, \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho}, 0\right)$ for some $c, d \in \mathbb{R}^n$ with $|c|^2 = |d|^2 = \rho\chi$ and $c \neq \pm d$.*

The admissible segments defined above correspond to suitable plane-wave solutions of (2.13). The following Lemma ensures that, for any $\rho, \chi \in \mathbb{R}^+$, the hyperinterior of $K^{co}_{\rho,\chi}$ is " sufficiently round " with respect to the special directions: given any point in the hyperinterior of $K^{co}_{\rho,\chi}$, it can be seen as the midpoint of a sufficiently large admissible segment for (ρ, χ).

Lemma 2.3.5. *There exists a constant $F = F(n) > 0$ such that for any $\rho, \chi \in \mathbb{R}^+$ and for any $z = (m, U, q) \in$ hint $K^{co}_{\rho,\chi}$ there exists an admissible line segment for (ρ, χ)*

$$\sigma = \left[(m, U, q) - (\overline{m}, \overline{U}, 0), (m, U, q) + (\overline{m}, \overline{U}, 0)\right] \tag{2.24}$$

such that

$$|\overline{m}| \geq \frac{F}{\sqrt{\rho\chi}} \left(\rho\chi - |m|^2\right).$$

The proof rests on a clever application of Carathéodory's theorem for convex sets and can be carried out, with minor modifications, as in [**DLS10**] (cf. Lemma 6 therein).

As an easy consequence of the previous Lemma, we can finally establish that the Λ-convex hull of $K_{\rho,\chi}$ coincides with $K^{co}_{\rho,\chi}$.

Proposition 2.3.6. *For all given $\rho, \chi \in \mathbb{R}^+$, the Λ-convex hull of $K_{\rho,\chi}$ coincides with the convex hull of $K_{\rho,\chi}$.*

Proof. Recall that, given $\rho, \chi \in \mathbb{R}^+$, we denote the Λ-convex hull of $K_{\rho,\chi}$ with $K^{\Lambda}_{\rho,\chi}$. Of course $K^{\Lambda}_{\rho,\chi} \subset K^{co}_{\rho,\chi}$, hence we have to prove the opposite inclusion, i.e. $K^{co}_{\rho,\chi} \subset K^{\Lambda}_{\rho,\chi}$. For every $z \in K^{co}_{\rho,\chi}$ we can follow the procedure in the proof of Lemma 2.3.5 (cf. [**DLS10**]) and write it as $z = \sum_j \lambda_j z_j$, with $(z_j)_{1\le j\le N+1}$ in $K_{\rho,\chi}$, $(\lambda_j)_{1\le j\le N+1}$ in $[0,1]$ and $\sum_j \lambda_j = 1$. Again, we can assume that $\lambda_1 = \max_j \lambda_j$. In case $\lambda_1 = 1$ then $z = z_1 \in K_{\rho,\chi} \subset K^{\Lambda}_{\rho,\chi}$ and we can already conclude. Otherwise (i.e. when $\lambda_1 \in (0,1)$) we can argue as in Lemma 2.3.5 so to find an admissible segment σ for (ρ,χ) of the form (2.24). Since we aim at writing z as a Λ-barycenter of elements of $K_{\rho,\chi}$, we "play" with these admissible segments by prolongations and iterative constructions until we get segments with extremes lying in $K_{\rho,\chi}$. More precisely: we extend the segment σ until we meet ∂ hint$K^{co}_{\rho,\chi}$ thus obtaining z as the barycenter of two points (w_0, w_1) with $(w_0 - w_1) \in \Lambda$ and such that every $w_i = (m_i, U_i, q_i)$, $i = 0, 1$, satisfies either $|m_i|^2 = \rho\chi$ or $|m_i|^2 < \rho\chi$ and $e(\rho, m_i, U_i) = \chi/n$.

In the first case, $U_i - \left(\frac{m_i \otimes m_i}{\rho} - \frac{|m_i|^2}{n\rho} I_n\right) \ge 0$, and since it is a null-trace-matrix it is identically zero, whence $w_i \in K_{\rho,\chi}$ $\Big($note that in the construction of σ the q-direction remains unchanged, hence $q_i = p(\rho) + \frac{\chi}{n}\Big)$.

In the second case, i.e. when $|m_i|^2 < \rho\chi$ and $e(\rho, m_i, U_i) = \chi/n$, we apply again Lemma 2.3.5 and a limit procedure to express w_i as barycentre of $(w_{i,0}, w_{i,1})$ with $(w_{i,0} - w_{i,1}) \in \Lambda$ and such that every $w_{i,k} = (m_{i,k}, U_{i,k}, q_{i,k})$, $k = 0, 1$, will satisfy either $|m_{i,k}|^2 = \rho\chi$ or $\lambda_2(\rho, m_{i,k}, U_{i,k}) = e(\rho, m_{i,k}, U_{i,k}) = \chi/n$, where $\lambda_1(\rho, m, U) \ge \lambda_2(\rho, m, U) \ge \ldots\ldots \ge \lambda_n(\rho, m, U)$ denote the ordered eigenvalues of the matrix $\frac{m\otimes m}{\rho} - U$ (note that $\lambda_1(\rho, m, U) = e(\rho, m, U)$). Now, we iterate this procedure of constructing suitable admissible segments for (ρ,χ) until we have written z as Λ-barycenter of points (m, U, q) satisfying either $|m|^2 = \rho\chi$ or $\lambda_n(\rho, m, U) = \chi/n$ and therefore all belonging to $K_{\rho,\chi}$ as desired. $\square$

2.4. A criterion for the existence of infinitely many solutions

The following Proposition provides a criterion to recognize initial data m^0 which allow for many weak admissible solutions to (2.1). Its proof relies deeply on the geometrical analysis carried out in Section

2.3. The underlying idea comes from convex integration. The general principle of this method, developed for partial differential equations by Gromov [**Gro86**] and for ordinary differential equations by Filippov [**Fil67**], consists in the following steps (cf. with Section 1.3.1.2): given a nonlinear equation $\mathcal{E}(z)$,

- (i) we rewrite it as $(\mathcal{L}(z) \wedge z \in K)$ where $\mathcal{L}$ is a linear equation;
- (ii) we introduce a strict subsolution z_0 of the system, i.e. satisfying a relaxed system $(\mathcal{L}(z_0) \wedge z_0 \in \mathcal{U})$;
- (iii) we construct a sequence $(z_k)_{k\in\mathbb{N}}$ approaching K but staying in $\mathcal{U}$;
- (iv) we pass to the limit, possibly modifying the sequence (z_k) in order to ensure a suitable convergence.

Step (i) has already been done in Section 2.1. The choice of z_0 will be specified in Sections 2.7-2.8. Here, we define the notion of subsolution for an appropriate set $\mathcal{U}$, we construct an improving sequence and we pass to the limit. The way how we construct the approximating sequence will be described in Section 2.6 using some tools from Section 2.5.

One crucial step in convex integration is the passage from open sets K to general sets. This can be done in different ways, e.g. by the Baire category theorem (cf. [**Oxt90**]), a refinement of it using Baire-1 functions or the Banach-Mazur game [**Kir03**] or by direct construction [**Syc01**]. Whatever approach one uses the basic theme is the same: at each step of the construction one adds a highly oscillatory correction whose frequency is much larger and whose amplitude is much smaller than those of the previous corrections.

In this section, we achieve our goals following some Baire category arguments as in [**DLS09**]: they are morally close to the methods developed by Bressan and Flores in [**BF94**], by Dacorogna and Marcellini in [**DM97**] and by Kirchheim in [**Kir03**] (see Section 1.3.1.1).

In our framework the initial data will be constructed starting from solutions to the convexified (or relaxed) problem associated to (2.8), i.e. solutions to the linearized system (2.13) satisfying a "relaxed" nonlinear constraint (2.14) (i.e. belonging to the hyperinterior of the convex hull of the "constraint set"), which we will call *subsolutions*.

As in [**DLS09**], our application shows that the Baire theory is comparable in terms of results to the method of convex integration and they have many similarities: they are both based on an approximation approach to tackle problems while the difference lies only in the limit arguments, i.e. on the way the exact solution is obtained from better and better approximate ones. These similarities are clarified by Kirchheim in [**Kir03**], where the continuity points of a first category Baire function are considered; a comparison between the two methods is drawn by Sychev in [**Syc01**].

Here, the topological reasoning of Baire theory is preferred to the iteration technique of convex integration, since the first has the advantage to provide us directly with infinitely many different solutions.

PROPOSITION 2.4.1. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function and let T be any finite positive time. Assume there exist (m_0, U_0, q_0) continuous space-periodic solutions of (2.13) on $\mathbb{R}^n \times]0,T[$ with*

$$m_0 \in C([0,T]; H_w(Q)), \tag{2.25}$$

and a function $\chi \in C^\infty([0,T];\mathbb{R}^+)$ such that

$$e(\rho_0(x), m_0(x,t), U_0(x,t)) < \frac{\chi(t)}{n} \qquad \text{for all } (x,t) \in \mathbb{R}^n \times]0,T[, \tag{2.26}$$

$$q_0(x,t) = p(\rho_0(x)) + \frac{\chi(t)}{n} \qquad \text{for all } (x,t) \in \mathbb{R}^n \times]0,T[. \tag{2.27}$$

Then there exist infinitely many weak solutions (ρ, m) of the system (2.8) in $\mathbb{R}^n \times [0,T[$ with density $\rho(x) = \rho_0(x)$ and such that

$$m \in C([0,T]; H_w(Q)), \tag{2.28}$$

$$m(\cdot,t) = m_0(\cdot,t) \qquad \text{for } t = 0,T \text{ and for a.e. } x \in \mathbb{R}^n, \tag{2.29}$$

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times]0,T[. \tag{2.30}$$

2.4.1. The space of subsolutions. We define the space of *subsolutions* as follows. Let ρ_0 and χ be given as in the assumptions of Proposition 2.4.1. Let m_0 be a vector field as in Proposition 2.4.1 with associated modified pressure q_0 and consider space-periodic momentum fields $m : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ which satisfy

$$\operatorname{div} m = 0, \tag{2.31}$$

the initial and boundary conditions

$$m(x,0) = m_0(x,0), \tag{2.32}$$

$$m(x,T) = m_0(x,T), \tag{2.33}$$

$$\tag{2.34}$$

and such that there exists a continuous space-periodic matrix field $U : \mathbb{R}^n \times]0,T[\to \mathcal{S}_0^n$ with

$$\begin{aligned} &e(\rho_0(x), m(x,t), U(x,t)) < \frac{\chi}{n} \quad \text{for all } (x,t) \in \mathbb{R}^n \times]0,T[, \\ &\partial_t m + \operatorname{div}_x U + \nabla_x q_0 = 0 \quad \text{in } \mathbb{R}^n \times [0,T]. \end{aligned} \tag{2.35}$$

Definition 2.4.2. *Let X_0 be the set of such linear momentum fields, i.e.*

$$X_0 = \Big\{ m \in C^0(]0,T[; C_p^0(Q)) \cap C([0,T]; H_w(Q)) :$$

$$(2.31) - (2.35) \textit{ are satisfied} \Big\} \tag{2.36}$$

and let X be the closure of X_0 in $C([0,T]; H_w(Q)$. Then X_0 will be the space of strict subsolutions.

As ρ_0 is continuous and periodic on $\mathbb{R}^n$ and χ is smooth on $[0,T]$, there exists a constant G such that $\chi(t) \int_Q \rho_0(x)dx \leq G$ for all $t \in [0,T]$. Since for any $m \in X_0$ with associated matrix field U we have that (see Lemma 2.3.2- (ii))

$$\begin{aligned} \int_Q |m(x,t)|^2 \, dx &\leq \int_Q n\rho_0(x) e(\rho_0(x), m(x,t), U(x,t)) dx \\ &< \chi(t) \int_Q \rho_0(x) dx \text{ for all } t \in [0,T], \end{aligned}$$

we can observe that X_0 consists of functions $m : [0,T] \to H(Q)$ taking values in a bounded subset B of $H(Q)$. Without loss of generality, we can assume that B is weakly closed. Then, B in its weak topology is metrizable and, if we let d_B be a metric on B inducing the weak topology, we have that (B, d_B) is a compact metric space. Moreover, we can define on $Y := C([0,T], (B, d_B))$ a metric d naturally induced by d_B via

$$d(f_1, f_2) := \max_{t \in [0,T]} d_B(f_1(\cdot,t), f_2(\cdot,t)). \tag{2.37}$$

Note that the topology induced on Y by d is equivalent to the topology of Y as a subset of $C([0,T];H_w)$. In addition, the space (Y,d) is complete. Finally, X is the closure in (Y,d) of X_0 and hence (X,d) is as well a complete metric space.

Lemma 2.4.3. *If $m \in X$ is such that $|m(x,t)|^2 = \rho_0(x)\chi(t)$ for almost every $(x,t) \in \mathbb{R}^n \times]0,T[$, then the pair (ρ_0, m) is a weak solution of* (2.8) *in $\mathbb{R}^n \times [0,T[$ satisfying* (2.28)-(2.29)-(2.30).

Proof. Let $m \in X$ be such that $|m(x,t)|^2 = \rho_0(x)\chi(t)$ for almost every $(x,t) \in \mathbb{R}^n \times]0,T[$. By density of X_0, there exists a sequence $\{m_k\} \subset X_0$ such that $m_k \stackrel{d}{\to} m$ in X. For any $m_k \in X_0$ let U_k be the associated smooth matrix field enjoying (2.35). Thanks to Lemma 2.3.2 (iii) and (2.35), the following pointwise estimate holds for the sequence $\{U_k\}$

$$|U_k|_\infty \leq (n-1)e(\rho_0, m_k, U_k) < \frac{(n-1)-\chi}{n}.$$

As a consequence, $\{U_k\}$ is uniformly bounded in $L^\infty([0,T];L^\infty_p(Q))$; by possibly extracting a subsequence, we have that

$$U_k \stackrel{*}{\rightharpoonup} U \text{ in } L^\infty([0,T];L^\infty_p(Q)).$$

Note that $\overline{\mathrm{hint} K^{co}_{\rho_0,\chi}} = K^{co}_{\rho_0,\chi}$ is a convex and compact set by Lemma 2.3.2-(i)-(ii)-(iii). Hence, $m \in X$ with associated matrix field U solves (2.13) on $\mathbb{R}^n \times [0,T]$ for $q = q_0$ and (m, U, q_0) takes values in $K^{co}_{\rho_0,\chi}$ almost everywhere. If, in addition, $|m(x,t)|^2 = \rho_0(x)\chi(t)$, then $(m, U, q_0)(x,t) \in K_{\rho,\chi}$ a.e. in $\mathbb{R}^n \times [0,T]$ (cf. Lemma 2.3.2-(iv)). Lemma 2.3.1 allows us to conclude that (ρ_0, m) is a weak solution of (2.8) in $\mathbb{R}^n \times [0,T[$. Finally, since $m_k \to m$ in $C([0,T];H_w(Q))$ and $|m(x,t)|^2 = \rho_0(x)\chi(t)$ for almost every $(x,t) \in \mathbb{R}^n \times]0,T[$, we see that m satisfies also (2.28)-(2.29)-(2.30). $\square$

Now, we will argue as in [**DLS09**] exploiting Baire category techniques to combine weak and strong convergence (see also [**Kir03**]).

Lemma 2.4.4. *The identity map $I : (X,d) \to L^2([0,T];H(Q))$ defined by $m \to m$ is a Baire-1 map, and therefore the set of points of continuity is residual in (X,d).*

Proof. Let $\phi_r(x,t) = r^{-(n+1)}\phi(rx,rt)$ be any regular spacetime convolution kernel. For each fixed $m \in X$, we have

$$\phi_r * m \to m \quad \text{strongly in } L^2(H) \text{ as } r \to 0.$$

On the other hand, for each $r > 0$ and $m_k \in X$,

$$m_k \xrightarrow{d} m \text{ implies } \phi_r * m_k \to \phi_r * m \text{ in } L^2(H).$$

Therefore, each map $I_r : (X, d) \to L^2(H)$, $m \to \phi_r * m$ is continuous, and $I(m) = \lim_{r\to 0} I_r(m)$ for all $m \in X$. This shows that $I : (X, d) \to L^2(H)$ is a pointwise limit of continuous maps; hence it is a Baire-1 map. As a consequence, the set of points of continuity of I is residual in (X, d) (cf. [**Oxt90**]). $\square$

2.4.2. Proof of Proposition 2.4.1. We aim to show that all points of continuity of the identity map correspond to solutions of (2.8) enjoying the requirements of Proposition 2.4.1: Lemma 2.4.4 will then allow us to prove Proposition 2.4.1 once we know that the cardinality of X is infinite. In light of Lemma 2.4.3, for our purposes it suffices to prove the following claim:

CLAIM. If $m \in X$ is a point of continuity of I, then

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \text{ for almost every } (x,t) \in \mathbb{R}^n \times]0, T[. \tag{2.38}$$

$\square$

Note that proving (2.38) is equivalent to prove that $\|m\|_{L^2(Q\times[0,T])} = \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx\right)^{1/2}$, since for any $m \in X$ we have $|m(x,t)|^2 \le \rho_0(x)\chi(t)$ for almost all $(x,t) \in \mathbb{R}^n \times [0,T]$. Thanks to this remark, the claim is reduced to the following lemma (cf. Lemma 4.6 in [**DLS09**]), which provides a strategy to move towards the boundary of X_0: given $m \in X_0$, we will be able to approach it with a sequence inside X_0 but closer than m to the boundary of X_0.

LEMMA 2.4.5. *Let ρ_0, χ be given functions as in Proposition 2.4.1. Then, there exists a constant $\beta = \beta(n)$ such that, given $m \in X_0$, there exists a sequence $\{m_k\} \subset X_0$ with the following properties*

$$\begin{aligned} \|m_k\|^2_{L^2(Q\times[0,T])} \ge{}& \|m\|^2_{L^2(Q\times[0,T])} \\ &+ \beta\left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx - \|m\|^2_{L^2(Q\times[0,T])}\right)^2 \end{aligned} \tag{2.39}$$

and

$$m_k \to m \text{ in } C([0,T], H_w(Q)). \tag{2.40}$$

The proof is postponed to Section 2.6. Let us show how Lemma 2.4.5 implies the claim. As in the claim, assume that $m \in X$ is a point of continuity of the identity map I. Let $\{m_k\} \subset X_0$ be a fixed sequence that converges to m in $C([0,T], H_w(Q))$. Using Lemma 2.4.5 and a standard diagonal argument, we can find a second sequence $\{\widetilde{m}_k\}$ yet converging to m in X and satisfying

$$\liminf_{k\to\infty} \|\widetilde{m}_k\|^2_{L^2(Q\times[0,T])} \geq \liminf_{k\to\infty} \Bigg(\|m_k\|^2_{L^2(Q\times[0,T])} + \beta \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx - \|m_k\|^2_{L^2(Q\times[0,T])} \right)^2 \Bigg).$$

According to the hypothesis, I is continuous at m, therefore both m_k and $\widetilde{m}_k$ converge strongly to m and

$$\|m\|^2_{L^2(Q\times[0,T])} \geq \|m\|^2_{L^2(Q\times[0,T])} + \beta \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx - \|m\|^2_{L^2(Q\times[0,T])} \right)^2 .$$

Hence $\|m\|_{L^2(Q\times[0,T])} = \left(\int_Q \int_0^T \rho_0(x)\chi(t)dtdx\right)^{1/2}$ and the claim holds true. Finally, since the assumptions of Proposition 2.4.1 ensure that X_0 is nonempty, by Lemma 2.4.5 we can see that the cardinality of X is infinite whence the cardinality of any residual set in X is infinite. In particular, the set of continuity points of I is infinite: this and the claim conclude the proof of Proposition 2.4.1.

2.5. Localized oscillating solutions

The wild solutions are made by adding one dimensional oscillating functions in different directions $\lambda \in \Lambda$. For that it is needed to localize the waves. More precisely, the proof of Lemma 2.4.5 relies on the construction of solutions to the linear system (2.13), localized in space-time and oscillating between two states in $K^{co}_{\rho_0,\chi}$ along a given special direction $\lambda \in \Lambda$. Aiming at compactly supported solutions, one faces the problem of localizing vector valued functions: this is bypassed thanks to the construction of a "localizing" potential for the conservation laws (2.13). This approach is inherited from [**DLS10**]. As in [**DLS09**] it could be realized for every $\lambda \in \Lambda$, but in our framework it is convenient to restrict only to *special* Λ-directions (cf. [**DLS10**]): this restriction will allow us to localize the oscillations at constant pressure.

Why oscillations at constant pressure are meaningful for us and needed in the proof of Lemma 2.4.5?

Owing to Section 2.3, in the variables $y = (x, t) \in \mathbb{R}^{n+1}$, the system (2.13) is equivalent to $\operatorname{div}_y M = 0$, where $M \in \mathcal{S}^{n+1}$ is defined via the linear map

$$\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} \ni (m, U, q) \longmapsto M = \begin{pmatrix} U + qI_n & m \\ m & 0 \end{pmatrix}. \tag{2.41}$$

More precisely, this map builds an identification between the set of solutions (m, U, q) to (2.13) and the set of symmetric $(n+1) \times (n+1)$ matrices M with $M_{(n+1)(n+1)} = 0$ and $\operatorname{tr}(M) = q$.

Therefore, solutions of (2.13) with $q \equiv 0$ correspond to matrix fields $M : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$\operatorname{div}_y M = 0, \quad M^{\mathrm{T}} = M, \quad M_{(n+1)(n+1)} = 0, \quad \operatorname{tr}(M) = 0. \tag{2.42}$$

Moreover, given a density ρ and two states $(c, U_c, q_c), (d, U_d, q_d) \in K_\rho$ with non collinear momentum vector fields c and d having same magnitude ($|c| = |d|$), and hence same pressure ($q_c = q_d$), then the corresponding matrices M_c and M_d have the following form

$$M_c = \begin{pmatrix} \frac{c\otimes c}{\rho} + p(\rho)I_n & c \\ c & 0 \end{pmatrix} \text{ and } M_d = \begin{pmatrix} \frac{d\otimes d}{\rho} + p(\rho)I_n & d \\ d & 0 \end{pmatrix}$$

and satisfy

$$M_c - M_d = \begin{pmatrix} \frac{c\otimes c}{\rho} - \frac{d\otimes d}{\rho} & c - d \\ c - d & 0 \end{pmatrix}.$$

Finally note that $\operatorname{tr}(M_c - M_d) = 0$ and $M_c - M_d \in \Lambda$ corresponds to a *special direction*.

The following Proposition provides a potential for solutions of (2.13) oscillating between two states M_c and M_d at constant pressure. It is an easy adaptation to our framework of Proposition 4 in [**DLS10**].

PROPOSITION 2.5.1. *Let $c, d \in \mathbb{R}^n$ such that $|c| = |d|$ and $c \neq d$. Let also $\rho \in \mathbb{R}$. Then there exists a matrix-valued, constant coefficient, homogeneous linear differential operator of order* 3

$$A(\partial) : C_c^\infty(\mathbb{R}^{n+1}) \to C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{(n+1)\times(n+1)})$$

such that $M = A(\partial)\phi$ satisfies (2.42) *for all $\phi \in C_c^\infty(\mathbb{R}^{n+1})$. Moreover there exists $\eta \in \mathbb{R}^{n+1}$ such that*

- *η is not parallel to e_{n+1};*
- *if $\phi(y) = \psi(y \cdot \eta)$, then*

$$A(\partial)\phi(y) = (M_c - M_d)\psi'''(y \cdot \eta).$$

We also report Lemma 7 from [**DLS10**]: it ensures that the oscillations of the planewaves generated in proposition 2.5.1 have a certain size in terms of an appropriate norm-type-functional.

LEMMA 2.5.2. *Let $\eta \in \mathbb{R}^{n+1}$ be a vector which is not parallel to e_{n+1}. Then for any bounded open set $B \subset \mathbb{R}^n$*

$$\lim_{N\to\infty} \int_B \sin^2(N\eta \cdot (x,t))dx = \frac{1}{2}\,|B|$$

uniformly in $t \in \mathbb{R}$.

For the proof we refer the reader to [**DLS10**].

2.6. The improvement step

We are now about to prove one of the cornerstones of the construction. Before moving forward, let us resume the plan. We have already identified a relaxed problem by introducing *subsolutions*. Then, we have proved a sort of "h-principle" (even if there is no homotopy here) according to which, the space of *subsolutions* can be "reduced" to the space of solutions or, equivalently, the typical (in Baire's sense) *subsolution* is a solution. Once assumed that a *subsolution* exists, the proof of our "h-principle" builds upon Lemma 2.4.5 combined with Baire category arguments. Indeed, we could also prove Proposition 2.4.1 by applying iteratively Lemma 2.4.5 and thus constructing a converging sequence of subsolutions approaching $K_{\rho,\chi}$: this would correspond to the constructive convex integration approach (see Section 1.3.1). So two steps are left in order to conclude our argument: showing the existence of a "starting" *subsolution* and prove Lemma 2.4.5.

This section is devoted to the second task, the proof of Lemma 2.4.5, while in next section we will exhibit a "concrete" *subsolution.*

What follows will be quite technical, therefore we first would like to recall the plan: we will add fast oscillations in allowed directions so to let $|m|^2$ increase in average. The proof is inspired by [**DLS09**]-[**DLS10**].

Proof. [Proof of Lemma 2.4.5] Let us fix the domain $\Omega := Q \times [0,T]$. We look for a sequence $\{m_k\} \subset X_0$, with associated matrix fields $\{U_k\}$, which improves m in the sense of (2.39) and has the form

$$(m_k, U_k) = (m, U) + \sum_j (\widetilde{m}_{k,j}, \widetilde{U}_{k,j}) \tag{2.43}$$

where every $z_{k,j} = (\widetilde{m}_{k,j}, \widetilde{U}_{k,j})$ is compactly supported in some suitable ball $B_{k,j}(x_{k,j}, t_{k,j}) \subset \Omega$. We proceed as follows.

Step 1. Let $m \in X_0$ with associated matrix field U. By Lemma 2.3.5, for any $(x,t) \in \Omega$ we can find a line segment

$$\sigma_{(x,t)} := [(m(x,t), U(x,t), q_0(x)) - \lambda_{(x,t)}, (m(x,t), U(x,t), q_0(x)) + \lambda_{(x,t)}]$$

admissible for $(\rho_0(x), \chi(t))$ and with direction

$$\lambda_{(x,t)} = (\overline{m}(x,t), \overline{U}(x,t), 0)$$

such that

$$|\overline{m}(x,t)| \geq \frac{F}{\sqrt{\rho_0(x)\chi(t)}} \left(\rho_0(x)\chi(t) - |m(x,t)|^2\right). \tag{2.44}$$

Since $z := (m,U)$ and $K^{co}_{\rho_0,\chi}$ are uniformly continuous in (x,t), there exists an $\varepsilon > 0$ such that for any $(x,t), (x_0,t_0) \in \Omega$ with $|x - x_0| + |t - t_0| < \varepsilon$, we have

$$(z(x,t), q_0(x)) \pm (\overline{m}(x_0,t_0), \overline{U}(x_0,t_0), 0) \subset \operatorname{hint} K^{co}_{\rho_0,\chi}. \tag{2.45}$$

Step 2. Fix $(x_0,t_0) \in \Omega$ for the moment. Now, let $0 \leq \phi_{r_0} \leq 1$ be a smooth cutoff function on Ω with support contained in a ball $B_{r_0}(x_0,t_0) \subset \Omega$ for some $r_0 > 0$, identically 1 on $B_{r_0/2}(x_0,t_0)$ and strictly less than 1 outside. Thanks to Proposition 2.5.1 and the identification $(m,U,q) \to M$, for the admissible line segment $\sigma_{(x_0,t_0)}$, there exist an operator A_0 and a direction $\eta_0 \in \mathbb{R}^{n+1}$ not parallel to e_{n+1}, such that for any $k \in \mathbb{N}$

$$A_0\left(\frac{\cos(k\eta_0 \cdot (x,t))}{k^3}\right) = \lambda_{(x_0,t_0)} \sin(k\eta_0 \cdot (x,t)),$$

and such that the pair $(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})$ defined by

$$(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})(x,t) := A_0\left[\phi_{r_0}(x,t) k^{-3} \cos(k\eta_0 \cdot (x,t))\right]$$

satisfies (2.13) with $q \equiv 0$. Note that $(\widetilde{m}_{k,0}, \widetilde{U}_{k,0})$ is supported in the ball $B_{r_0}(x_0, t_0)$ and that

$$\begin{aligned} &\left\| (\widetilde{m}_{k,0}, \widetilde{U}_{k,0}) - \phi_{r_0} \left(\overline{m}(x_0,t_0), \overline{U}(x_0,t_0) \right) \sin(k\eta_0 \cdot (x,t)) \right\|_\infty \\ &\leq \mathit{const}\, (A_0, \eta_0, \|\phi_0\|_{C^3})\, \frac{1}{k} \end{aligned} \tag{2.46}$$

since A_0 is a linear differential operator of homogeneous degree 3. Furthermore, for all $(x,t) \in B_{r_0/2}(x_0,t_0)$, we have

$$|\widetilde{m}_{k,0}(x,t)|^2 = |\overline{m}(x_0,t_0)|^2 \sin^2(k\eta_0 \cdot (x,t)).$$

Since $\eta_0 \in \mathbb{R}^{n+1}$ is not parallel to e_{n+1}, from Lemma 2.5.2 we can see that

$$\lim_{k\to\infty} \int_{B_{r_0/2}(x_0,t_0)} |\widetilde{m}_{k,0}(x,t)|^2 \, dx = \frac{1}{2} \int_{B_{r_0/2}(x_0,t_0)} |\overline{m}(x_0,t_0)|^2 \, dx$$

uniformly in t. In particular, using (2.44), we obtain

$$\begin{aligned} &\lim_{k\to\infty} \int_{B_{r_0/2}(x_0,t_0)} |\widetilde{m}_{k,0}(x,t)|^2 \, dxdt \geq \\ &\frac{F^2}{2\rho_0(x_0)\chi(t_0)} \left(\rho_0(x_0)\chi(t_0)) - |m(x_0,t_0)|^2 \right)^2 \left| B_{r_0/2}(x_0,t_0) \right|. \end{aligned} \tag{2.47}$$

Step 3. Next, observe that since m is uniformly continuous, there exists an $\bar{r} > 0$ such that for any $r < \bar{r}$ there exists a finite family of pairwise disjoint balls $B_{r_j}(x_j,t_j) \subset \Omega$ with $r_j < r$ such that

$$\begin{aligned} &\int_\Omega \left(\rho_0(x)\chi(t) - |m(x,t)|^2 \right)^2 dxdt \leq \\ &2 \sum_j \left(\rho_0(x_j)\chi(t_j) - |m(x_j,t_j)|^2 \right)^2 \left| B_{r_j}(x_j,t_j) \right|. \end{aligned} \tag{2.48}$$

Fix $s > 0$ with $s < \min\{\bar{r}, \varepsilon\}$ and choose a finite family of pairwise disjoint balls $B_{r_j}(x_j,t_j) \subset \Omega$ with radii $r_j < s$ such that (2.48) holds. In each ball $B_{2r_j}(x_j,t_j)$ we apply the construction of *Step 2* to obtain, for every $k \in \mathbb{N}$, a pair $(\widetilde{m}_{k,j}, \widetilde{U}_{k,j})$.

Final step. Letting (m_k, U_k) be as in (2.43), we observe that the sum therein consists of finitely many terms. Therefore from (2.45) and (2.46) we deduce that there exists $k_0 \in \mathbb{N}$ such that

$$m_k \in X_0 \text{ for all } k \geq k_0. \tag{2.49}$$

Moreover, owing to (2.47) and (2.48) we can write

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |m_k(x,t)-m(x,t)|^2\,dxdt &= \lim_{k\to\infty}\sum_j\int_\Omega |\widetilde{m}_{k,j}(x,t)|^2\,dxdt\\ &\geq \sum_j \frac{F^2}{2\rho_0(x_j)\chi(t_j)}\left(\rho_0(x_j)\chi(t_j)) - |m(x_j,t_j)|^2\right)^2 \left|B_{r_j}(x_j,t_j)\right|\\ &\geq C\int_\Omega \left(\rho_0(x)\chi(t)-|m(x,t)|^2\right)^2 dxdt. \qquad (2.50)\end{aligned}$$

Since $m_k \overset{d}{\rightarrow} m$, due to (2.50) we have

$$\begin{aligned}\liminf_{k\to\infty}\|m_k\|^2_{L^2(\Omega)} &= \|m\|_2^2 + \liminf_{k\to\infty}\|m_k-m\|_2^2\\ &\geq \|m\|_2^2 + C\int_\Omega \left(\rho_0(x)\chi(t)-|m(x,t)|^2\right)^2 dxdt, \qquad (2.51)\end{aligned}$$

which gives (2.39) with $\beta=\beta(n)=\beta(F(n))$. □

2.7. Construction of suitable initial data

In this section we show the existence of a *subsolution* in the sense of Definition 2.4.2. Since the subsolution we aim to construct has to be space-periodic, it will be enough to work on the building brick Q and then extend the construction periodically to $\mathbb{R}^n$.

The idea to work in the space-periodic setting has been recently adopted by Wiedemann [**Wie11**] in order to construct global solutions to the incompressible Euler equations, i.e. to prove Theorem 1.3.9.

PROPOSITION 2.7.1. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function as in Proposition 2.4.1 and let T be any given positive time. Then, there exist a smooth function $\widetilde{\chi}:\mathbb{R}\to\mathbb{R}^+$, a continuous periodic matrix field $\widetilde{U}:\mathbb{R}^n\to\mathcal{S}^n_0$ and a function $\widetilde{q}\in C^1(\mathbb{R};C^1_p(\mathbb{R}^n))$ such that*

$$\operatorname{div}_x\widetilde{U}+\nabla_x\widetilde{q}=0 \qquad \text{on} \quad \mathbb{R}^n\times\mathbb{R} \qquad (2.52)$$

and

$$e(\rho_0(x),0,\widetilde{U}(x)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t)\in\mathbb{R}^n\times[0,T[\qquad (2.53)$$

$$\widetilde{q}(x,t) = p(\rho_0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } x\in\mathbb{R}^n\times\mathbb{R}. \qquad (2.54)$$

Proof. [Proposition 2.7.1] Let us define $\widetilde{U}$ componentwise by its Fourier transform as follows:

$$\begin{aligned}\widehat{\widetilde{U}}_{ij}(k) &:= \left(\frac{nk_ik_j}{(n-1)\,|k|^2}\right)\widehat{p(\rho_0(k))} \text{ if } i \neq j,\\ \widehat{\widetilde{U}}_{ii}(k) &:= \left(\frac{nk_i^2-|k|^2}{(n-1)\,|k|^2}\right)\widehat{p(\rho_0(k))}.\end{aligned} \tag{2.55}$$

for every $k \neq 0$, and $\widehat{\widetilde{U}}(0) = 0$. Clearly $\widehat{\widetilde{U}}_{ij}$ thus defined is symmetric and trace-free. Moreover, since $p(\rho_0) \in C^1_p(\mathbb{R}^n)$, standard elliptic regularity arguments allow us to conclude that $\widetilde{U}$ is a continuous periodic matrix field. Next, notice that

$$\left\|e(\rho_0(x), 0, \widetilde{U}(x))\right\|_\infty = \left\|\lambda_{max}(-\widetilde{U})\right\|_\infty = \widetilde{\lambda} \tag{2.56}$$

for some positive constant $\widetilde{\lambda}$. Therefore, we can choose any smooth function $\widetilde{\chi}$ on $\mathbb{R}$ such that $\widetilde{\chi} > n\widetilde{\lambda}$ on $[0,T]$ in order to ensure (2.53). Now, let $\widetilde{q}$ be defined exactly as in (2.54) for the choice of $\widetilde{\chi}$ just done. It remains to show that (2.52) holds. In light of (2.54), we can write equation (2.52) in Fourier space as

$$\sum_{j=1}^{n} k_j\widehat{\widetilde{U}}_{ij} = k_i\widehat{p(\rho_0)} \tag{2.57}$$

for $k \in \mathbb{Z}^n$. It is easy to check that $\widehat{\widetilde{U}}$ as defined by (2.55) solves (2.57) and hence $\widetilde{U}$ and $\widetilde{q}$ satisfy (2.52)

□

REMARK 2.7.2. *We note that the Hölder continuity of ρ_0 would be enough to argue as in the previous proof in order to infer the continuity of $\widetilde{U}$.*

PROPOSITION 2.7.3. *Let $\rho_0 \in C^1_p(Q;\mathbb{R}^+)$ be a given density function as in Proposition 2.4.1 and let T be any given positive time. There exist triples $(\overline{m}, \overline{U}, \overline{q})$ solving (2.13) distributionally on $\mathbb{R}^n \times \mathbb{R}$ enjoying the* *following properties:*

$$(\overline{m}, \overline{U}, \overline{q}) \text{ is continuous in } \mathbb{R}^n \times (\mathbb{R}\backslash\{0\}) \text{ and } \overline{m} \in C(\mathbb{R}; H_w(\mathbb{R}^n)), \tag{2.58}$$

$$\overline{U}(\cdot, t) = \widetilde{U}(\cdot) \text{ for } t = -T, T \tag{2.59}$$

and

$$\overline{q}(x) = p(\rho_0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.60}$$

$$e(\rho_0(x), \overline{m}(x,t), \overline{U}(x,t)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times ([-T,0[\cap]0,T]). \tag{2.61}$$

Moreover

$$|\overline{m}(x,0)|^2 = \rho_0(x)\chi(0) \text{ a.e. in } \mathbb{R}^n. \tag{2.62}$$

Proof. [Proposition 2.7.3] We first choose $\overline{q} := \tilde{q}$ given by Proposition 2.7.1. This choice already yields (2.60).

Now, in analogy with Definition 2.4.2 we consider the space X_0 defined as the set of continuous vector fields $m : \mathbb{R}^n \times]-T,T[\to \mathbb{R}^n$ in $C^0(]-T,T[; C^0_p(Q))$ to which there exists a continuous space-periodic matrix field $U : \mathbb{R}^n \times]-T,T[\to \mathcal{S}^n_0$ such that

$$\begin{aligned} \operatorname{div}_x m &= 0, \\ \partial_t m + \operatorname{div}_x U + \nabla_x \overline{q} &= 0, \end{aligned} \tag{2.63}$$

$$\operatorname{supp}(m) \subset Q \times [-T/2, T/2[\tag{2.64}$$

$$U(\cdot,t) = \widetilde{U}(\cdot) \text{ for all } t \in [-T,T[\backslash[-T/2,T/2] \tag{2.65}$$

and

$$e(\rho_0(x), m(x,t), U(x,t)) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times]-T,T[. \tag{2.66}$$

As in Section 2.4.1, X_0 consists of functions $m :]-T,T[\to H$ taking values in a bounded set $B \subset H$. On B the weak topology of L^2 is metrizable, and correspondingly we find a metric d on $C(]-T,T[; B)$ inducing the topology of $C(]-T,T[; H_w(Q))$.

Next we note that with minor modifications the proof of Lemma 2.4.5 leads to the following claim:

Claim: Let $Q_0 \subset Q$ be given. Let $m \in X_0$ with associated matrix field U and let $\alpha > 0$ such that

$$\int_{Q_0} \left[|m(x,0)|^2 - (\rho_0(x)\widetilde{\chi}(0))\right] dx < -\alpha$$

Then, for any $\delta > 0$ there exists a sequence $m_k \in X_0$ with associated smooth matrix field U_k such that

$$\operatorname{supp}(m_k - m, U_k - U) \subset Q_0 \times [-\delta, \delta],$$

$$m_k \overset{d}{\to} m,$$

and

$$\liminf_{k\to\infty} \int_{Q_0} |m_k(x,0)|^2 \geq \int_{Q_0} |m(x,0)|^2 \, dx + \beta\alpha^2.$$

Fix an exhausting sequence of bounded open subsets $Q_k \subset Q_{k+1} \subset Q$, each compactly contained in Ω, and such that $|Q_{k+1} \backslash Q_k| \leq 2^{-k}$. Let also γ_ε be a standard mollifying kernel in $\mathbb{R}^n$ (the unusual notation γ_ε for the standard mollifying kernel is aimed at avoiding confusion between it and the density function). Using the claim above we construct inductively a sequence of momentum vector fields $m_k \in X_0$, associated matrix fields U_k and a sequence of numbers $\eta_k < 2^{-k}$ as follows.

First of all let $m_1 \equiv 0$, $U_1(x,t) = \widetilde{U}(x)$ for all $(x,t) \in \mathbb{R}^{n+1}$ and having obtained (m_1, U_1), ..., (m_k, U_k), $\eta_1, ..., \eta_{k-1}$ we choose $\eta_k < 2^{-k}$ in such a way that

$$\|m_k - m_k * \gamma_{\eta_k}\|_{L^2} < 2^{-k}. \tag{2.67}$$

Then, we set

$$\alpha_k = -\int_{Q_k} [|m_k(x,0)|^2 - \rho_0(x)\widetilde{\chi}(0))]dx.$$

Note that (2.66) ensures $\alpha_k > 0$. Then, we apply the claim with Q_k, $\alpha = \alpha_k$ and $\delta = 2^{-k}T$ to obtain $m_{k+1} \in X_0$ and associated smooth matrix field U_{k+1} such that

$$\operatorname{supp}(m_{k+1} - m_k, U_{k+1} - U_k) \subset Q_k \times [-2^{-k}T, 2^{-k}T], \tag{2.68}$$

$$d(m_{k+1}, m_k) < 2^{-k}, \tag{2.69}$$

$$\int_{Q_k} |m_{k+1}(x,0)|^2\,dx \geq \int_{Q_k} |m_k(x,0)|^2\,dx + \beta\alpha_k^2. \tag{2.70}$$

Since d induces the topology of $C(]-T,T[;H_w(\Omega))$ we can also require that

$$\left\|(m_k - m_{k+1}) * \gamma_{\eta_j}\right\|_{L^2(\Omega)} < 2^{-k} \text{ for all } j \leq k \text{ for } t = 0. \tag{2.71}$$

From (2.7) we infer the existence of a function $\overline{m} \in C(]-T,T[,H_w(\Omega))$ such that

$$m_k \xrightarrow{d} \overline{m}.$$

Besides, (2.68) implies that for any compact subset S of $Q\times]-T,0[\cup]0,T[$ there exists k_0 such that $(m_k,U_k)|_S = (m_{k_0},U_{k_0})|_S$ for all $k > k_0$. Hence (m_k,U_k) converges in $C^0_{\text{loc}}(Q\times]-T,0[\cup]0,T[)$ to a continuous pair $(\overline{m},\overline{U})$ solving equations (2.63) in $\mathbb{R}^n\times]-T,0[\cup]0,T[$ and such that (2.58)-(2.61) hold. In order to conclude, we show that also (2.62) holds for $\overline{m}$.

As first, we observe that (2.70) yields

$$\alpha_{k+1} \leq \alpha_k - \beta\alpha_k^2 + |Q_{k+1}\backslash Q_k| \leq \alpha_k - \beta\alpha_k^2 + 2^{-k},$$

from which we deduce that

$$\alpha_k \to 0 \text{ as } k \to \infty.$$

This, together with the following inequality

$$0 \geq \int_Q \left[|m_k(x,0)|^2 - \rho_0(x)\chi(0)\right] dx \geq -(\alpha_k + C\,|Q\backslash Q_k|) \geq -(\alpha_k + C2^{-k}),$$

implies that

$$\lim_{k\uparrow\infty} \int_\Omega \left[|m_k(x,0)|^2 - \rho_0(x)\chi(0)\right] dx = 0. \tag{2.72}$$

On the other hand, owing to (2.67) and (2.71), we can write for $t = 0$ and for every k

$$\begin{aligned}
&\|m_k - \overline{m}\|_{L^2} \\
&\leq \|m_k - m_k * \gamma_{\eta_k}\|_{L^2} + \|m_k * \gamma_{\eta_k} - \overline{m} * \gamma_{\eta_k}\|_{L^2} + \|\overline{m} * \gamma_{\eta_k} - \overline{m}\|_{L^2} \\
&\leq 2^{-k} + \sum_{j=0}^{\infty} \|m_{k+j} * \gamma_{\eta_k} - m_{k+j+1} * \gamma_{\eta_k}\|_{L^2} + 2^{-k} \\
&\leq 2^{-(k-2)}.
\end{aligned} \tag{2.73}$$

Finally, (2.73) implies that $m_k(\cdot, 0) \to \overline{m}(\cdot, 0)$ strongly in $H(Q)$ as $k \to \infty$, which together with (2.72) gives

$$|\overline{m}(x,0)|^2 = \rho_0(x)\chi(0) \text{ for almost every } x \in \mathbb{R}^n.$$

□

2.8. Proof of the main Theorems

Proof. [Proof of Theorem 2.2.1] Let T be any finite positive time and $\rho_0 \in C^1_p(Q)$ be a given density function. Let also $(\overline{m}, \overline{U}, \overline{q})$ be as in Proposition 2.7.3. Then, define $\chi(t) := \widetilde{\chi}(t)$, $q_0(x) := \overline{q}(x)$,

$$m_0(x,t) = \begin{cases} \overline{m}(x,t) \text{ for } t \in [0,T] \\ \overline{m}(x, t-2T) \text{ for } t \in [T, 2T], \end{cases} \tag{2.74}$$

$$U_0(x,t) = \begin{cases} \overline{U}(x,t) \text{ for } t \in [0,T] \\ \overline{U}(x, t-2T) \text{ for } t \in [T, 2T]. \end{cases} \tag{2.75}$$

For this choices, the quadruple (m_0, U_0, q_0, χ) satisfies the assumptions of Proposition 2.4.1. Therefore, there exist infinitely many solutions $m \in C([0, 2T], H_w(Q))$ of (2.8) in $\mathbb{R}^n \times [0, 2T[$ with density ρ_0, such that

$$m(x,0) = \overline{m}(x,0) = m(x,2T) \text{ for a.e. } x \in \Omega$$

and

$$|m(\cdot,t)|^2 = \rho_0(\cdot)\chi(0) \text{ for almost every } (x,t) \in \mathbb{R}^n \times]0, 2T[. \tag{2.76}$$

Since $|m_0(\cdot,0)|^2 = \rho_0(\cdot)\chi(0)$ a.e. in $\mathbb{R}^n$ as well, it is enough to define $m^0(x) = m_0(x,0)$ to satisfy also (2.12) and hence conclude the proof.

□

Proof. [Proof of Theorem 2.2.2] Under the assumptions of Theorem 2.2.1, we have proven the existence of a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0,T]; H_w(Q))$ of (2.8) on $\mathbb{R}^n \times [0,T[$ with density ρ_0. Moreover, the proof (see Proof of Proposition 2.7.1) showed that for any smooth function $\chi : \mathbb{R} \to \mathbb{R}^+$ with $\chi > n\widetilde{\lambda} > 0$ the following holds

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0,T[, \tag{2.77}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n. \tag{2.78}$$

Now, we claim that there exist constants $C_1, C_2 > 0$ such that choosing the function $\chi(t) > n\widetilde{\lambda}$ on $[0,T[$ among solutions of the following differential inequality

$$\chi'(t) \leq -C_1\chi^{1/2}(t) - C_2\chi^{3/2}(t), \tag{2.79}$$

then the weak solutions (ρ_0, m) of (2.8) obtained in Theorem 2.2.1 will also satisfy the admissibility condition (2.7) on $\mathbb{R}^n \times [0,T[$. Of course, there is an issue of compatibility between the differential inequality (2.79) and the condition $\chi > n\widetilde{\lambda}$: this motivates the existence of a time $\overline{T} > 0$ defining the maximal time-interval in which the admissibility condition indeed holds.

Let T be any finite positive time. As first, we aim to prove the claim. Since $m \in C([0,T]; H_w(Q))$ is divergence-free and fulfills (2.77)-(2.78) and ρ_0 is time-independent, (2.7) reduces to the following inequality

$$\frac{1}{2}\chi'(t) + m \cdot \nabla\left(\varepsilon(\rho_0(x)) + \frac{p(\rho_0(x))}{\rho_0(x)}\right) + \frac{\chi(t)}{2} m \cdot \nabla\left(\frac{1}{\rho_0(x)}\right) \leq 0, \tag{2.80}$$

intended in the sense of (space-periodic) distributions on $\mathbb{R}^n \times [0,T]$. As $\rho_0 \in C^1_p(Q)$, there exists a constant c_0^2 with $\rho_0 \leq c_0^2$ on $\mathbb{R}^n$, whence (see (2.77)-(2.78))

$$|m(x,t)| \leq c_0\sqrt{\chi(t)} \text{ a.e. on } \mathbb{R}^n \times [0,T[. \tag{2.81}$$

Similarly we can find constants $c_1, c_2 > 0$ with

$$\left|\nabla\left(\varepsilon(\rho_0(x)) + \frac{p(\rho_0(x))}{\rho_0(x)}\right)\right| \leq c_1 \text{ a.e. in } \mathbb{R}^n \tag{2.82}$$

$$\left|\nabla\left(\frac{1}{\rho_0(x)}\right)\right| \leq c_2 \text{ a.e. in } \mathbb{R}^n. \tag{2.83}$$

As a consequence of (2.81)-(2.83), (2.80) holds as soon as χ satisfies

$$\chi'(t) \leq -2c_1c_0\chi^{1/2}(t) - c_2c_0\chi^{3/2}(t) \text{ on } [0, T[.$$

Therefore, by choosing $C_1 := 2c_1c_0$ and $C_2 := c_2c_0$ we can conclude the proof of the claim.

Now, it remains to show the existence of a function χ as in the claim, i.e. that both the differential inequality (2.79) and the condition $\chi > n\widetilde{\lambda}$ can hold true on some suitable time-interval. To this aim, we can consider the equality in (2.79), couple it with the initial condition $\chi(0) = \chi_0$ for some constant $\chi_0 > n\widetilde{\lambda}$ and then solve the resulting Cauchy problem. For the obtained solution χ, there exists a positive time $\overline{T}$ such that $\chi(t) > n\widetilde{\lambda}$ on $[0, \overline{T}[$.

Finally, applying the claim on the time-interval $[0, \overline{T}[$ we conclude that the admissibility condition holds on $\mathbb{R}^n \times [0, \overline{T}[$ as desired. $\square$

Proof. [Proof of Theorem 2.1.1] The proof of Theorem 2.1.1 strongly relies on Theorems 2.2.1-2.2.2. Given a continuously differentiable initial density ρ^0 we apply Theorems 2.2.1-2.2.2 for $\rho_0(x) := \rho^0(x)$ thus obtaining a positive time $\overline{T}$ and a bounded initial momentum m^0 allowing for infinitely many solutions $m \in C([0, T]; H_w(Q))$ of (2.8) on $\mathbb{R}^n \times [0, \overline{T}[$ with density ρ^0 and such that the following holds

$$|m(x,t)|^2 = \rho_0(x)\chi(t) \quad \text{a.e. in } \mathbb{R}^n \times [0, \overline{T}[, \tag{2.84}$$

$$\left|m^0(x)\right|^2 = \rho_0(x)\chi(0) \quad \text{a.e. in } \mathbb{R}^n, \tag{2.85}$$

for a suitable smooth function $\chi : [0, \overline{T}] \to \mathbb{R}^+$. Now, define $\rho(x, t) = \rho_0(x)\mathbf{1}_{[0,\overline{T}[}(t)$. This shows that (2.6) holds. To prove (2.5) observe that ρ is independent of t and m is weakly divergence-free for almost every $0 < t < \overline{T}$. Therefore, the pair (ρ, m) is a weak solution of (2.1) with initial data (ρ^0, m^0). Finally, we can also prove (2.7): each solution obtained is also admissible. Indeed, for $\rho(x, t) = \rho_0(x)\mathbf{1}_{[0,\overline{T}[}(t)$, (2.7) is ensured by Theorem 2.2.2. $\square$

CHAPTER 3

Global ill-posedness of the isentropic system of gas dynamics with regular initial data

In this chapter, we will focus on the Cauchy problem for the isentropic compressible Euler equations in two space dimensions: we will show that even for some smooth initial data non–uniqueness of bounded admissible solutions arises after the first blow–up time (see [**CDL**]). The chapter is mainly devoted to the proof of the following surprising theorem which corresponds to Theorem 0.2.2 in the Introduction of the thesis.

THEOREM 3.0.1. *There are Lipschitz initial data* (ρ^0, v^0) *for which there are infinitely many bounded admissible solutions* (ρ, v) *of the isentropic compressible Euler equations on* $\mathbb{R}^2 \times [0, \infty[$ *with* $\inf \rho > 0$.

Theorem 3.0.1 is achieved by showing the existence of classical Riemann data (i.e. pure jump discontinuities across a line) which can be generated by a compression wave and for which there are infinitely many bounded admissible solutions of the isentropic compressible Euler equations. In the next section, we will clarify the main ideas behind the proof of Theorem 3.0.1 and outline the structure of the chapter.

3.1. Introduction

We consider the Cauchy problem for the isentropic compressible Euler equations in two space dimensions. This system (i.e. system (0.1)) consists of 3 scalar equations, which state the conservation of mass and linear momentum. The unknowns are the density ρ and the velocity v. We recall that the resulting Cauchy problem takes the form:

$$
\begin{cases}
\partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\
\partial_t(\rho v) + \operatorname{div}_x\left(\rho v \otimes v\right) + \nabla_x[p(\rho)] \;=\; 0 \\
\rho(\cdot, 0) \;=\; \rho^0 \\
v(\cdot, 0) \;=\; v^0 \,.
\end{cases}
\tag{3.1}
$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas under consideration and it is assumed to satisfy $p' > 0$ (this hypothesis guarantees also the hyperbolicity of the system on the regions where ρ is positive). A common choice is the polytropic pressure law $p(\rho) = \kappa\rho^\gamma$ with constants $\kappa > 0$ and $\gamma > 1$. The classical kinetic theory of gases predicts exponents $\gamma = 1 + \frac{2}{d}$, where d is the degree of freedom of the molecule of the gas. Here we will be concerned mostly with the particular choice $p(\rho) = \rho^2$. However several technical statements hold under the general assumption $p' > 0$ and the specific choice $p(\rho) = \rho^2$ is relevant only to some portions of the proofs in this chapter.

In this chapter we show that, in more than one space dimension, the most popular concept of admissible solution fails to give uniqueness even under some very strong assumptions on the initial data. In particular we consider bounded weak solutions of (3.1), i.e. satisfying (3.1) in the usual distributional sense (we refer to Definition 3.2.1), and we call them admissible if they satisfy the following additional inequality in the sense of distibutions (called usually *entropy inequality*, although for the specific system (3.1) this is rather a weak form of the energy balance):

$$\partial_t \left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2}\right) + \mathrm{div}_x \left[\left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho)\right) v\right] \le 0 \tag{3.2}$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2\varepsilon'(r)$. Indeed, admissible solutions are required to satisfy a slightly stronger condition, i.e. a form of (3.2) which involves also the initial data, cp. with Definition 3.2.2. For all the solutions considered in this chapter ρ will always be bounded away from 0, i.e. there will be a positive constant c_0 such that $\rho \ge c_0$.

We denote the space variable as $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{3.3}$$

where $\rho_\pm, v_\pm$ are constants. It is well-known that for some special choices of these constants there are solutions of (3.1) which are *rarefaction waves*, i.e. self-similar solutions depending only on t and x_2

which are locally Lipschitz for positive t and constant on lines emanating from the origin (see [**Daf10**, Section 7.6]). Reversing their order (i.e. exchanging $+$ and $-$) the very same constants allow for a *compression wave* solution, i.e. a solution on $\mathbb{R}^2 \times (-\infty, 0)$ which is locally Lipschitz and converges, for $t \uparrow 0$, to the jump discontinuity of (3.3). When this is the case we will then say that the data (3.3) is *generated by a classical compression wave.*

We are now ready to give the precise statements corresponding to Theorem 3.0.1. The main theorem is the following:

THEOREM 3.1.1. *Assume $p(\rho) = \rho^2$. Then there are Riemann data data as in* (3.3) *for which there are infinitely many bounded admissible solutions (ρ, v) of* (3.1) *on $\mathbb{R}^2 \times [0, \infty[$ with $\inf \rho > 0$. These data are indeed all generated by classical compression waves.*

It follows from the usual treatment of the 1-dimensional Riemann problem that for the data of Theorem 3.1.1 uniqueness holds if the admissible solutions are also required to be self-similar, i.e. of the form $(\rho, v)(x, t) = \left(r\left(\frac{x_2}{t}\right), w\left(\frac{x_2}{t}\right)\right)$ and to have locally bounded variation (see indeed Chapter 4). Note that such solutions must be discontinuous, because the data of Theorem 3.1.1 are generated by compression waves. We in fact conjecture that this is the case for *any* initial data (3.3) allowing the nonuniqueness property of Theorem 3.1.1: however this fact does not seem to follow from the usual weak-strong uniqueness (as for instance in [**Daf10**, Theorem 5.3.1]) because the Lipschitz constant of the classical solution blows up as $t \downarrow 0$.

As an obvious corollary of Theorem 3.1.1 we obtain Theorem 3.0.1:

COROLLARY 3.1.2. *There are Lipschitz initial data (ρ^0, v^0) for which there are infinitely many bounded admissible solutions (ρ, v) of* (3.1) *on $\mathbb{R}^2 \times [0, \infty[$ with $\inf \rho > 0$. These solutions are all locally Lipschitz on a finite interval of time where they therefore all coincide with the unique classical solution.*

3.1.1. h-principle and the Euler equations. The proof of Theorem 3.1.1 builds heavily upon the works of De Lellis and Székelyhidi [**DLS09**]–[**DLS10**], who had already observed that the methods developed to explain the existence of compactly supported nontrivial weak solutions of the *incompressible* Euler equations could be exported to the compressible Euler equations and lead to the ill-posedness of bounded admissible solutions, see [**DLS10**]. However, the data of [**DLS10**] were extremely irregular and left the question whether the ill-posedness was not due to the irregularity of the data, rather than to the irregularity of the solution. A first answer is provided in the work [**Chi11**], here explained in Chapter 2, where we showed that data with very regular densities but irregular velocities still allow for nonuniqueness of admissible solutions.

In this chapter we give a complete answer to this question since even for some smooth initial data nonuniqueness of bounded admissible solutions arises after the first blow-up time. The results here presented were recently obtained in [**CDL**]. It remains however an open question how irregular the solutions have to be in order to display the pathological behaviour of Theorem 3.1.1. One could speculate that, in analogy to what has been shown recently for the incompressible Euler equations, even a "piecewise Hölder regularity" might not be enough; see [**DLS12**], [**DLSZ12**], [**BDS13**] and in particular [**DA13**].

This is also inspired from the work [**Sz11**] where Székelyhidi coupled the methods introduced in [**DLS09**]-[**DLS10**] with a clever construction to produce rather surprising irregular solutions of the incompressible Euler equations with vortex-sheet initial data. This work of Székelyhidi was in turn motivated by the so-called Muskat problem (see [**CFG09**], [**Szé11**] and [**Shv11**]), we moreover refer to [**DLS11**] for a rather detailed survey).

Especially relevant for us is the appropriate notion of *subsolution*, which allows to use the methods of [**DLS09**]-[**DLS10**] to solve the equations *and* impose a certain specific initial data. We refer again to [**DLS11**] for the motivation behind this concept and its link to existing literature in physics and mathematics, whereas here we only show how Theorem 3.1.1 can be reduced to construct solutions of an appropriate (larger) system of PDEs coupled with some algebraic constraints: such solutions will be called *fan admissible subsolutions*, cp. with Definitions 3.2.4 and 3.2.5. In Section 3.3, by making some specific choices, the existence of this subsolution is reduced to finding an array of real numbers satisfying some algebraic identities and inequalities, cp. with Proposition 3.3.1. So far all the statements can be carried out for a general pressure law p.

Although there seems to be a certain abundance of solutions to this set of identities and inequalities, we do not have currently an efficient and general method to find them. We propose two possible ways in the Sections 3.4 and 3.5. The one of Section 3.4 is the most effective and produces the initial data of Theorem 3.1.1 which are generated by a compression wave. The one of Section 3.5 is an alternative strategy where, instead of making a precise choice of the pressure law p, we exploit it as an extra degree of freedom: as a result this method gives data as in Theorem 3.1.1 but with a different pressure law, which is essentially a suitable smoothing of the step-function. We also do not know whether the latter data can be generated by compression waves.

Finally, in Section 3.6 we give an alternative proof of Székelyhidi's result [**Sz11**] on the vortex–sheet problem of incompressible fluid dynamics.

3.2. Subsolutions

3.2.1. Weak and admissible solutions of (3.1)**.** We recall here the usual definitions of weak and admissible solutions to (3.1).

Definition 3.2.1. *By a weak solution of* (3.1) *on* $\mathbb{R}^2 \times [0,\infty[$ *we mean a pair* $(\rho, v) \in L^\infty(\mathbb{R}^2 \times [0,\infty[)$ *such that the following identities hold for every test functions* $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty[)$, $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty[)$*:*

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho \partial_t \psi + \rho v \cdot \nabla_x \psi]\, dxdt + \int_{\mathbb{R}^2} \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{3.4}$$

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} &[\rho v \cdot \partial_t \phi + \rho v \otimes v : D_x \phi + p(\rho)\,\mathrm{div}_x\, \phi] \\ &+ \int_{\mathbb{R}^2} \rho^0(x) v^0(x) \cdot \phi(x,0) dx \;=\; 0. \end{aligned} \tag{3.5}$$

Definition 3.2.2. *A bounded weak solution* (ρ, v) *of* (3.1) *is admissible if it satisfies the following inequality for every nonnegative test function* $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty[)$*:*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^2} \left[\left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi + \left(\rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] \\ &\quad + \int_{\mathbb{R}^2} \left(\rho^0 \varepsilon(\rho^0) + \rho^0 \frac{|v^0|^2}{2} \right) \varphi(\cdot, 0) \;\geq\; 0\,. \end{aligned} \tag{3.6}$$

3.2.2. Subsolutions. To start with, we make precise the definition of subsolution in our context. Here $\mathcal{S}_0^{2\times 2}$ denotes the set of symmetric traceless 2×2 matrices and Id is the identity matrix. We first introduce a notion of good partition for the upper half-space $\mathbb{R}^2\times]0,\infty[$.

Definition 3.2.3 (Fan partition). *A* fan partition *of* $\mathbb{R}^2\times]0,\infty[$ *consists of finitely many open sets* $P_-, P_1, \dots P_N, P_+$ *of the following form*

$$P_- = \{(x,t) : t>0 \quad \text{and} \quad x_2 < \nu_- t\} \tag{3.7}$$

$$P_+ = \{(x,t) : t>0 \quad \text{and} \quad x_2 > \nu_+ t\} \tag{3.8}$$

$$P_i = \{(x,t) : t>0 \quad \text{and} \quad \nu_{i-1} t < x_2 < \nu_i t\} \tag{3.9}$$

where $\nu_- = \nu_0 < \nu_1 < \ldots < \nu_N = \nu_+$ *is an arbitrary collection of real numbers.*

Definition 3.2.4 (Fan Compressible subsolutions). *A fan subsolution to the compressible Euler equations* (3.1) *with initial data* (3.3) *is a triple* $(\overline{\rho}, \overline{v}, \overline{u}) : \mathbb{R}^2\times]0,\infty[\to (\mathbb{R}^+, \mathbb{R}^2, \mathcal{S}_0^{2\times 2})$ *of bounded measurable functions satisfying the following requirements.*

(i) *There is a fan partition* $P_-, P_1, \dots, P_N, P_+$ *of* $\mathbb{R}^2\times]0,\infty[$ *such that*

$$(\overline{\rho}, \overline{v}, \overline{u}) = \sum_{i=1}^{N} (\rho_i, v_i, u_i)\mathbf{1}_{P_i} + (\rho_-, v_-, u_-)\mathbf{1}_{P_-} + (\rho_+, v_+, u_+)\mathbf{1}_{P_+}$$

where ρ_i, v_i, u_i *are constants with* $\rho_i > 0$ *and* $u_\pm = v_\pm \otimes v_\pm - \frac{1}{2}|v_\pm|^2\mathrm{Id}$*;*

(ii) *For every* $i \in \{1, \dots, N\}$ *there exists a positive constant* C_i *such that*

$$v_i \otimes v_i - u_i < \frac{C_i}{2}\mathrm{Id} \ \ a.e.. \tag{3.10}$$

(iii) *The triple* $(\overline{\rho}, \overline{v}, \overline{u})$ *solves the following system in the sense of distributions:*

$$\partial_t \overline{\rho} + \mathrm{div}_x(\overline{\rho}\,\overline{v}) \ = \ 0 \tag{3.11}$$

$$\partial_t(\overline{\rho}\,\overline{v}) + \mathrm{div}_x\,(\overline{\rho}\,\overline{u}) + \nabla_x \left(p(\overline{\rho}) + \frac{1}{2}\left(\sum_i C_i \rho_i \mathbf{1}_{P_i} + \overline{\rho}|\overline{v}|^2 \mathbf{1}_{P_+\cup P_-}\right)\right) = 0 \tag{3.12}$$

Definition 3.2.5 (Admissible fan subsolutions). *A fan subsolution* $(\overline{\rho}, \overline{v}, \overline{u})$ *is said to be* admissible *if it satisfies the following inequality in the sense of distributions*

$$\begin{aligned} \partial_t \left(\overline{\rho}\varepsilon(\overline{\rho})\right) &+ \operatorname{div}_x \left[\left(\overline{\rho}\varepsilon(\overline{\rho}) + p(\overline{\rho})\right) \overline{v}\right] \\ &+ \sum_{i=1}^{N} \left[\partial_t \left(\rho_i \frac{C_i}{2} \mathbf{1}_{P_i}\right) + \operatorname{div}_x \left(\rho_i\, v_i \frac{C_i}{2} \mathbf{1}_{P_i}\right)\right] \\ &+ \partial_t \left(\overline{\rho}\frac{|\overline{v}|^2}{2} \mathbf{1}_{P_+ \cup P_-}\right) + \operatorname{div}_x \left(\overline{\rho}\frac{|\overline{v}|^2}{2}\overline{v} \mathbf{1}_{P_+ \cup P_-}\right) \;\leq\; 0\,. \end{aligned} \tag{3.13}$$

It is possible to generalize these notions in several directions, e.g. allowing partitions with more general open sets and functions v_i, u_i and ρ_i which vary (for instance continuously) in each element of the partition. It is not difficult to extend the conclusions of the next subsection to such settings. However we have chosen to keep the definitions to the minimum needed for our proof of Theorem 3.1.1.

3.2.3. Reduction to admissible fan subsolutions.

Using the techniques introduced in [**DLS09**]-[**DLS10**] we then reduce Theorem 3.1.1 to finding an admissible fan subsolution through the following proposition.

Proposition 3.2.6. *Let p be any C^1 function and $(\rho_\pm, v_\pm)$ be such that there exists at least one admissible fan subsolution $(\overline{\rho}, \overline{v}, \overline{u})$. Then there are infinitely many bounded admissible solutions (ρ, v) to* (3.1)-(3.3) *such that $\rho = \overline{\rho}$.*

The core of the proof is in fact a corresponding statement for subsolutions of the *incompressible Euler* which is essentially contained in the proofs of [**DLS09**]-[**DLS10**]. However, since we need such statement with slightly different assumptions, we state it here more precisely.

Lemma 3.2.7. *Let $(\tilde{v}, \tilde{u}) \in \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2}$ and $C > 0$ be such that $\tilde{v} \otimes \tilde{v} - \tilde{u} < \frac{C}{2}\mathrm{Id}$. For any open set $\Omega \subset \mathbb{R}^2 \times \mathbb{R}$ there are infinitely many maps $(\underline{v}, \underline{u}) \in L^\infty(\Omega, \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2})$ with the following property*

(i) *$\underline{v}$ and $\underline{u}$ vanish identically outside Ω;*
(ii) *$\operatorname{div}_x \underline{v} = 0$ and $\partial_t \underline{v} + \operatorname{div}_x \underline{u} = 0$;*
(iii) *$(\tilde{v} + \underline{v}) \otimes (\tilde{v} \otimes \underline{v}) - (\tilde{u} + \underline{u}) = \frac{C}{2}\mathrm{Id}$ a.e. on Ω.*

The proof is a minor variant of the ones given in [**DLS09**]-[**DLS10**] but since none of the statements present in the literature matches exactly the one of Lemma 3.2.7 we give some of the details, reducing to precise lemmas in the papers [**DLS09**]-[**DLS10**].

Proof. We define X_0 to be the space of $(\underline{v}, \underline{u}) \in C^\infty_c(\Omega, \mathbb{R}^2 \times \mathcal{S}^{2\times 2}_0)$ which satisfy (ii) and the pointwise inequality $(\tilde{v} + \underline{v}) \otimes (\tilde{v} \otimes \underline{v}) - (\tilde{u} + \underline{u}) < \frac{C}{2}\mathrm{Id}$. We then take the closure X of X_0 in the L^∞ weak* topology and recall that, since X is a bounded (weakly*) closed subset of L^∞ such topology is metrizable on X, giving a complete metric space (X, d). Observe that any element in X satisfies (i) and (ii) and we want to show that on a residual set (in the sense of Baire category) (iii) holds. We then define for any $N \in \mathbb{N} \setminus \{0\}$ the map I_N as follows: to $(\underline{v}, \underline{u})$ we associate the corresponding restrictions of these maps to $B_N(0) \times]-N, N[$. We then consider I_N as a map from (X, d) to Y, where Y is the space $L^\infty(B_N(0) \times]-N, N[, \mathbb{R}^2 \times \mathcal{S}^{2\times 2}_0)$ endowed with the *strong* L^2 topology. Arguing as in [**DLS09**, Lemma 4.5] it is easily seen that I_N is a Baire-1 map and hence, from a classical theorem in Baire category, its points of continuity are a residual set in X. We claim that

(Cl) if $(\underline{v}, \underline{u})$ is a point of continuity of I_N, then (iii) holds a.e. on $B_N(0) \times]-N, N[$.

(Cl) implies then (iii) for those maps at which *all* I_N are continuous (which is also a residual set).

The proof of (Cl) is achieved as in [**DLS09**, Lemma 4.6] showing that, when (Cl) fails, there is a sequence (v_k, u_k) converging weakly* to $(\underline{v}, \underline{u})$ for which $\|\tilde{v}\mathbf{1}_\Gamma + v_k\|_{L^2(\Gamma)} \geq \|\tilde{v}\mathbf{1}_\Gamma + \underline{v}\|^2_{L^2(\Gamma)} + \beta(C|\Gamma| - \|\tilde{v}\mathbf{1}_\Gamma + \underline{v}\|^2_{L^2(\Gamma)})$, where $\Gamma = B_N(0) \times]-N, N[$. Indeed we would like to apply [**DLS09**, Lemma 4.6] to $(v_0, u_0, q_0) = (\tilde{v} + \underline{v}\mathbf{1}_\Gamma, \tilde{u} + \underline{u}\mathbf{1}_\Gamma, 0)$. There are however some modifications that must be applied:

- first of all, the [**DLS09**, Lemma 4.6] is stated in the case $C = 1$, but since we are dealing with linear differential constraints, we can easily reduce to this case by multiplying our maps with a constant factor $C^{-\frac{1}{2}}$; hence we assume w.l.o.g. $C = 1$; observe then that indeed the maps (v_0, u_0, q_0) take values in the set $\mathcal{U}$ of [**DLS09**, (13)] thanks to the explicit computation of the convex hull of the set K of [**DLS09**, (12)] as given in [**DLS10**, Lemma 3];

- secondly, observe that our maps (v_0, u_0, q_0) do not satisfy $\partial_t v_0 + \operatorname{div}_x u_0 + \nabla_x q_0 = 0$ and $\operatorname{div} v_0 = 0$ (as required by [**DLS09**, Lemma 4.6]), but they in fact satisfy $\partial_t(v_0 - \tilde{v}\mathbf{1}_\Gamma) + \operatorname{div}_x(u_0 - \tilde{u}\mathbf{1}_\Gamma) + \nabla_x q_0 = 0$ and $\operatorname{div}(v_0 - \tilde{v}\mathbf{1}_\Gamma) = 0$; however this does not play any role, since in fact the maps (v_k, u_k, q_k) of [**DLS09**, Lemma 4.6] are achieved by adding to (v_0, u_0, q_0) perturbations of the form (V_k, U_k, Q_k) with $\partial_t V_k + \operatorname{div}_x U_k + \nabla_x Q_k = 0$ and $\operatorname{div}_x V_k = 0$ and this strategy works obviously also in our case;
- finally we in fact would need a sequence of triples (V_k, U_k, Q_k) as above where Q_k vanishes identically; this can be easily achieved following the same arguments as in the proofs in [**DLS09**] but applying the potentials of [**DLS10**, Proposition 4] where instead we use the ones of [**DLS09**, Lemma 3.4].

$\square$

Proof. [Proof of Proposition 3.2.6] We apply Lemma 3.2.7 in each region $\Omega = P_i$ and we call $(\underline{v}_i, \underline{u}_i)$ any pair of maps given by such Lemma. Hence we set

$$v := \overline{v} + \sum_{i=1}^{N} \underline{v}_i \tag{3.14}$$

$$u := \overline{u} + \sum_{i=1}^{N} \underline{u}_i \tag{3.15}$$

whereas $\rho = \overline{\rho}$ (as claimed in the statement of the Proposition!). We next show that the pair (ρ, v) is an admissible weak solution of (3.1)-(3.3). First observe that $\operatorname{div}_x(\rho_i \underline{v}_i) = 0$ since ρ_i is a constant. But since $\underline{v}_i$ is supported in P_i and $\rho = \overline{\rho} \equiv \rho_i$ on P_i, we then conclude $\operatorname{div}_x(\overline{\rho}\underline{v}_i) = 0$. Thus, in the sense of distributions, we have

$$\begin{aligned}\partial_t \rho + \operatorname{div}_x(\rho v) &= \partial_t \overline{\rho} + \operatorname{div}_x \left(\overline{\rho v} + \sum_i \overline{\rho}\underline{v}_i \right) \\ &= \partial_t \overline{\rho} + \operatorname{div}_x(\overline{\rho v}) + \sum_i \operatorname{div}_x(\overline{\rho}\underline{v}_i) = \partial_t \overline{\rho} + \operatorname{div}_x(\overline{\rho v}) = 0\,.\end{aligned} \tag{3.16}$$

Moreover, observe that

$$v \otimes v = \begin{cases} v_+ \otimes v_+ & \text{on } P_+ \\ v_- \otimes v_- & \text{on } P_- \\ (v_i + \underline{v}_i) \otimes (v_i + \underline{v}_i) = u_i + \underline{u}_i + \frac{C_i}{2}\mathrm{Id} & \text{on } P_i \end{cases}$$

and

$$\overline{u} = \begin{cases} v_+ \otimes v_+ - \frac{1}{2}|v_+|^2 \mathrm{Id} & \text{on } P_+ \\ v_- \otimes v_- - \frac{1}{2}|v_-|^2 \mathrm{Id} & \text{on } P_- \\ u_i & \text{on } P_i \,. \end{cases}$$

Hence, we can write

$$\begin{aligned} &\partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] \\ =&\partial_t \left(\overline{\rho v} + \sum_i \rho_i \underline{v}_i \right) + \mathrm{div}_x \left(\overline{\rho u} + \sum_i \rho_i \underline{u}_i \right) \\ &+ \nabla_x \left(p(\overline{\rho}) + \frac{1}{2} \sum_i C_i \rho_i \mathbf{1}_{P_i} + \frac{1}{2}|v_-|^2 \rho_- \mathbf{1}_{P_-} + \frac{1}{2}|v_+|^2 \rho_+ \mathbf{1}_{P_+} \right) \\ =&\partial_t \left(\overline{\rho v} \right) + \mathrm{div}_x \left(\overline{\rho u} \right) \\ &+ \nabla_x \left(p(\overline{\rho}) + \frac{1}{2} \sum_i C_i \rho_i \mathbf{1}_{P_i} + \frac{1}{2}|v_-|^2 \rho_- \mathbf{1}_{P_-} + \frac{1}{2}|v_+|^2 \rho_+ \mathbf{1}_{P_+} \right) \\ &+ \sum_i \rho_i \underbrace{\partial_t \underline{v}_i + \mathrm{div}_x \, \underline{u}_i}_{=0} \,. \end{aligned} \tag{3.17}$$

Therefore, by Definition 3.2.4 we conclude $\partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] = 0$.

Next, we compute

$$\begin{aligned} &\partial_t \left(\rho \varepsilon(\rho) + \frac{|v|^2}{2} \rho \right) + \mathrm{div}_x \left(\left(\rho \varepsilon(\rho) + \frac{|v|^2}{2} \rho + p(\rho) \right) v \right) \\ =&\partial_t \left(\overline{\rho} \varepsilon(\overline{\rho}) + \sum_i \frac{1}{2} C_i \rho_i \mathbf{1}_{P_i} + \frac{|v_-|^2}{2} \rho_- \mathbf{1}_{P_-} + \frac{|v_+|^2}{2} \rho_+ \mathbf{1}_{P_+} \right) \\ &+ \mathrm{div}_x \left[\left(\overline{\rho} \varepsilon(\overline{\rho}) + p(\overline{\rho}) + \sum_i \frac{1}{2} C_i \rho_i \mathbf{1}_{P_i} + \frac{|v_-|^2}{2} \rho_- \mathbf{1}_{P_-} + \frac{|v_+|^2}{2} \rho_+ \mathbf{1}_{P_+} \right) \right. \\ &\left. \cdot \left(\overline{v} + \sum_i \underline{v}_i \right) \right] \end{aligned}$$

Using the condition (3.13) we therefore conclude

$$\partial_t\left(\rho\varepsilon(\rho)+\frac{|v|^2}{2}\rho\right)+\mathrm{div}_x\left(\left(\rho\varepsilon(\rho)+\frac{|v|^2}{2}\rho+p(\rho)\right)v\right)\leq$$
$$\sum_i \mathrm{div}_x\left[\underline{v}_i\underbrace{\Big(\overline{\rho}\varepsilon(\overline{\rho})+p(\overline{\rho})+\sum_i\frac{1}{2}C_i\rho_i\mathbf{1}_{P_i}+\frac{|v_-|^2}{2}\rho_-\mathbf{1}_{P_-}+\frac{|v_+|^2}{2}\rho_+\mathbf{1}_{P_+}\Big)}_{=:\varrho}\right]$$

in the sense of distributions. Observe however that the function ϱ is constant on each P_i, where $\underline{v}_i$ is supported. Thus

$$\partial_t\left(\rho\varepsilon(\rho)+\frac{|v|^2}{2}\rho\right)+\mathrm{div}_x\left(\left(\rho\varepsilon(\rho)+\frac{|v|^2}{2}\rho+p(\rho)\right)v\right)$$
$$\leq\sum_i\varrho\,\mathrm{div}_x\,\underline{v}_i=0\,.$$

Observe that so far we have shown that (3.4), (3.5) and (3.6) hold whenever the corresponding test functions are supported in $\mathbb{R}^2\times]0,\infty[$. However observe that, since as $\tau\downarrow 0$ the Lebesgue measure of $P_i\cap\{t=\tau\}$ converges to 0, the maps $\rho(\tau,\cdot)$ and $v(\tau,\cdot)$ converge to the maps ρ^0 and v^0 of (3.3) strongly in L^1_{loc}. This easily implies (3.4), (3.5) and (3.6) in their full generality. For instance, assume $\psi\in C^\infty_c(\mathbb{R}^2\times]-\infty,\infty[)$ and consider a smooth cut-off function ϑ of time only which vanishes identically on $]-\infty,\varepsilon]$ and is identically 1 on $]\delta,\infty[$, where $0<\varepsilon<\delta$. We know therefore that (3.4) holds for the test function $\psi\vartheta$, which implies that

$$\int_0^\infty\int_{\mathbb{R}^2}\vartheta\left[\rho\partial_t\psi+\rho v\cdot\nabla_x\psi\right]dxdt+\int_0^\delta\int_{\mathbb{R}^2}\vartheta'(t)\rho(x,t)\psi(x,t)dxdt=0\,.$$

Fix δ and let ϑ converge to the function

$$\eta(t)=\begin{cases}0 & \text{if } t\leq 0\\ 1 & \text{if } t\geq\delta\\ \frac{t}{\delta} & \text{if } 0\leq t\leq\delta.\end{cases}$$

so that also ϑ' converges pointwise to $\frac{1}{\delta}\mathbf{1}_{]0,\delta[}$. We then conclude

$$\int_0^\infty\int_{\mathbb{R}^2}\eta\left[\rho\partial_t\psi+\rho v\cdot\nabla_x\psi\right]dxdt+\frac{1}{\delta}\int_0^\delta\int_{\mathbb{R}^2}\rho(x,t)\psi(x,t)dxdt=0\,.$$

Letting $\delta\downarrow 0$ we conclude (3.4).

The remaining conditions (3.5) and (3.6) are achieved with analogous arguments, which we leave to the reader. □

3.3. A set of algebraic identities and inequalities

In this case we look at fan subsolutions with a fan partition consisting of only three sets, that are P_-, P_1 and P_+.

We introduce therefore the real numbers $\alpha, \beta, \gamma, \delta, v_{-1}, v_{-2}, v_{+1}, v_{+2}$ with the properties that

$$v_1 = (\alpha, \beta), \tag{3.18}$$

$$v_- = (v_{-1}, v_{-2}) \tag{3.19}$$

$$v_+ = (v_{+1}, v_{+2}) \tag{3.20}$$

$$u_1 = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix}. \tag{3.21}$$

PROPOSITION 3.3.1. *Let $N = 1$ and P_-, P_1, P_+ be a fan partition as in Definition 3.2.3. The constants $v_1, v_-, v_+, u_1, \rho_-, \rho_+, \rho_1$ as in* (3.18)-(3.21) *define an admissible fan subsolution as in Definitions 3.2.4-3.2.5 if and only if the following identities and inequalities hold:*

- *Rankine-Hugoniot conditions on the left interface*

$$\nu_-(\rho_- - \rho_1) = \rho_- v_{-2} - \rho_1 \beta \tag{3.22}$$

$$\nu_-(\rho_- v_{-1} - \rho_1 \alpha) = \rho_- v_{-1} v_{-2} - \rho_1 \delta \tag{3.23}$$

$$\nu_-(\rho_- v_{-2} - \rho_1 \beta) = \rho_-(v_{-2})^2 + \rho_1 \gamma + p(\rho_-) - p(\rho_1) - \rho_1 \frac{C_1}{2} \tag{3.24}$$

- *Rankine-Hugoniot conditions on the right interface*

$$\nu_+(\rho_1 - \rho_+) = \rho_1 \beta - \rho_+ v_{+2} \tag{3.25}$$

$$\nu_+(\rho_1 \alpha - \rho_+ v_{+1}) = \rho_1 \delta - \rho_+ v_{+1} v_{+2} \tag{3.26}$$

$$\nu_+(\rho_1 \beta - \rho_+ v_{+2}) = -\rho_1 \gamma - \rho_+(v_{+2})^2 + p(\rho_1) - p(\rho_+) + \rho_1 \frac{C_1}{2} \tag{3.27}$$

- *Subsolution condition*

$$\alpha^2 + \beta^2 < C_1 \tag{3.28}$$

$$\left(\frac{C_1}{2} - \alpha^2 + \gamma\right)\left(\frac{C_1}{2} - \beta^2 - \gamma\right) - (\delta - \alpha\beta)^2 > 0 \tag{3.29}$$

- *Admissibility condition on the left interface*

$$
\begin{aligned}
&\nu_-(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1)) + \nu_-\left(\rho_-\frac{|v_-|^2}{2} - \rho_1\frac{C_1}{2}\right) \\
&\leq [(\rho_-\varepsilon(\rho_-) + p(\rho_-))v_{-2} - (\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta] \\
&\quad + \left(\rho_- v_{-2}\frac{|v_-|^2}{2} - \rho_1\beta\frac{C_1}{2}\right)
\end{aligned}
\tag{3.30}
$$

- *Admissibility condition on the right interface*

$$
\begin{aligned}
&\nu_+(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+)) + \nu_+\left(\rho_1\frac{C_1}{2} - \rho_+\frac{|v_+|^2}{2}\right) \\
&\leq [(\rho_1\varepsilon(\rho_1) + p(\rho_1))\beta - (\rho_+\varepsilon(\rho_+) + p(\rho_+))v_{+2}] \\
&\quad + \left(\rho_1\beta\frac{C_1}{2} - \rho_+ v_{+2}\frac{|v_+|^2}{2}\right)
\end{aligned}
\tag{3.31}
$$

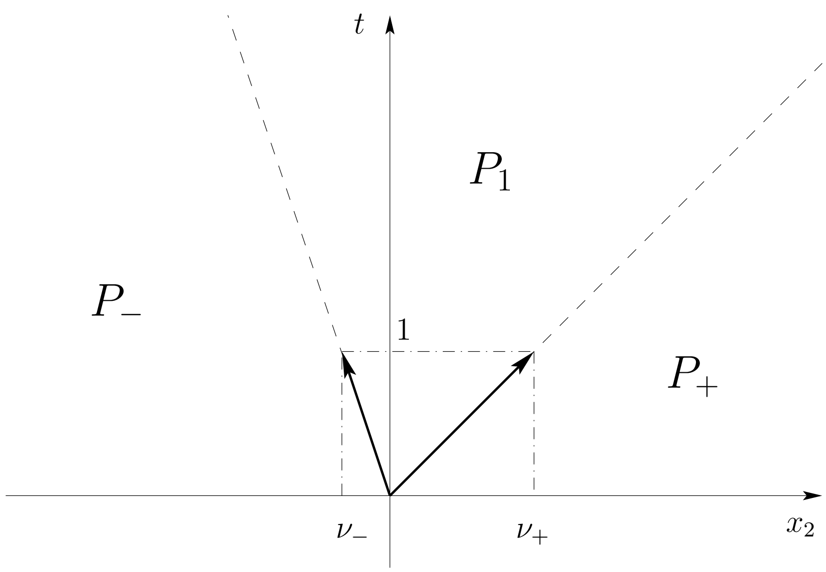

FIGURE 1. The fan partition in three regions.

Proof. Observe that the triple $(\overline{\rho}, \overline{v}, \overline{u})$ does not depend on the variable x_1. We will therefore consider it as a map defined on the t, x_2 plane. The various conditions and inequalities follow from straightforward computations, recalling that the maps $\overline{\rho}, \overline{v}$ and $\overline{u}$ are constant in the regions P_-, P_1 and P_+ shown in Figure 1. In particular

- The identities (3.22) and (3.25) are equivalent to the continuity equation (3.11), in particular they come from the corresponding "Rankine-Hugoniot" type conditions at the interfaces between P^- and P_1 (the *left interface*) and P_1 and P_+ (the *right interface*), respectively;
- The identitities (3.23) and (3.26) are the Rankine-Hugoniot conditions at the left and right interfaces coming from the first component of the momentum equation (3.12); similarly (3.24) and (3.27) correspond to the Rankine-Hugoniot conditions at the left and right interfaces for the second component of the momentum equation (3.12);
- The inequalities (3.28) and (3.29) are derived applying the usual criterion that the matrix

$$M := \frac{C_1}{2}\mathrm{Id} - v_1 \otimes v_1 + u_1 \tag{3.32}$$

  is positive definite if and only if $\operatorname{tr} M$ and $\det M$ are both positive;
- Finally, the conditions (3.30) and (3.31) come from the admissibility condition (3.13), again considering, respectively, the corresponding inequalities at the left and right interfaces.

□

3.4. First method: data generated by compression waves for $p(\rho) = \rho^2$

In this section we show how to find solutions of the algebraic constraints in Proposition 3.3.1 when $p(\rho) = \rho^2$ with pairs $(\rho_\pm, v_\pm)$ which can be connected by a compression wave, thereby showing Theorem 3.1.1. We start by recalling the following fact, which can be easily derived using (by now) standard theory of hyperbolic conservation laws in one space dimension.

Lemma 3.4.1. *Let* $0 < \rho_- < \rho_+$, $v_+ = (-\frac{1}{\rho_+}, 0)$ *and* $v_- = (-\frac{1}{\rho_+}, 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-}))$. *Then there is a pair* $(\rho, v) \in W^{1,\infty}_{loc} \cap L^\infty(\mathbb{R}^2 \times]-\infty, 0[, \mathbb{R}^+ \times \mathbb{R}^2)$ *such that*

(i) $\rho_+ \geq \rho \geq \rho_- > 0$;

(ii) *The pair solves the hyperbolic system*

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \; = \; 0 \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] \; = \; 0 \end{cases} \tag{3.33}$$

with $p(\rho) = \rho^2$ in the classical sense (pointwise a.e. and distributionally);

(iii) *for $t \uparrow 0$ the pair $(\rho(\cdot,t), v(\cdot,t))$ converges pointwise a.e. to (ρ^0, v^0) as in* (3.3)*;*

(iv) *$(\rho(\cdot,t), v(\cdot,t)) \in W^{1,\infty}$ for every $t < 0$.*

As already mentioned, the proof is a very standard application of the one-dimensional theory for the so-called Riemann problem. However, we give the details for the reader's convenience.

Proof. We look for solutions (ρ, v) as in the claim which are independent of the x_1 variable. Moreover we observe that, since we will produce classical $W^{1,\infty}_{loc}$ solutions, the admissibility condition (3.6) will be automatically satisfied as an equality because

$$\left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho, \left(\rho\varepsilon(\rho) + \frac{|v|^2}{2}\rho + p(\rho)\right) v\right)$$

is an entropy-entropy flux pair for the system (3.33) (cp. with [**Daf10**, Sections 3.2, 3.3.6, 4.1]). We then introduce the unknowns

$$(m_1(x_2,t), m_2(x_2,t)) = m(x_2,t) := v(x_2,t)\rho(x_2,t)$$

and hence rewrite the system as

$$\left\{\begin{array}{rcl} \partial_t\rho + \partial_{x_2} m_2 &=& 0 \\ \partial_t m_1 + \partial_{x_2}\left(\frac{m_1 m_2}{\rho}\right) &=& 0 \\ \partial_t m_2 + \partial_{x_2}\left(\frac{{m_2}^2}{\rho} + \rho^2\right) &=& 0 \end{array}\right. \tag{3.34}$$

Observe that if (ρ, m) is a solution of (3.34) then also

$$(\tilde\rho(x_2,t), \tilde m(x_2,t)) := (\rho(-x_2,-t), m(-x_2,-t))$$

is. Moreover, if (ρ, m) is locally Lipschitz and hence satisfies the admissibility condition with *equality*, so does $(\tilde\rho, \tilde m)$. We have therefore reduced ourselves to finding classical $W^{1,\infty}_{loc}$ solutions on $\mathbb{R}\times]0,\infty[$ of (3.34) with initial data

$$\rho_0(x) := \begin{cases} \rho_L & \text{if } x_2 < 0, \\ \rho_R & \text{if } x_2 > 0, \end{cases} \tag{3.35}$$

and

$$m_0(x) := \begin{cases} m_R := \left(-\frac{\rho_R}{\rho_L}, 2\sqrt{2}\rho_R(\sqrt{\rho_L} - \sqrt{\rho_R})\right) & \text{if } x_2 > 0, \\ m_L := (-1, 0) & \text{if } x_2 < 0, \end{cases} \tag{3.36}$$

where $\rho_+ = \rho_L > \rho_R = \rho_- > 0$, $m_L = v_+\rho_+$ and $m_R = v_-\rho_-$.

The problem amounts to show that, under our assumptions, there is a classical rarefaction wave solving, forward in time, the system (3.34) with initial data (ρ_0, m_0) as in (3.35) and (3.36). We set therefore $p(\rho) = \rho^2$ and we look for a locally Lipschitz self-similar solution (ρ, m) to the Riemann problem (3.34)-(3.35)-(3.36):

$$(\rho, m)(x_2, t) = (R, M)\left(\frac{x_2}{t}\right), \quad -\infty < x_2 < \infty, \quad 0 < t < \infty, \tag{3.37}$$

where (R, M) are locally Lipschitz functions on $(-\infty, \infty)$ which satisfy the ordinary differential equations

$$\begin{aligned}
&\frac{d}{d\xi}\left[M_2(\xi) - \xi R(\xi)\right] + R(\xi) = 0\\
&\frac{d}{d\xi}\left[\frac{M_1(\xi)M_2(\xi)}{R(\xi)} - \xi M_1(\xi)\right] + M_1(\xi) = 0\\
&\frac{d}{d\xi}\left[\frac{M_2(\xi)^2}{R(\xi)} + p(R(\xi)) - \xi M_2(\xi)\right] + M_2(\xi) = 0\,.
\end{aligned}$$

Before analyzing our specific Riemann problem, we review some general notions for system (3.34). If we define the state vector $U := (\rho, m_1, m_2)$, we can recast the system (3.34) in the general form

$$\partial_t U + \partial_{x_2} F(U) = 0,$$

where

$$F(U) := \begin{pmatrix} m_2 \\ \frac{m_1 m_2}{\rho} \\ \frac{{m_2}^2}{\rho} + p(\rho) \end{pmatrix}.$$

By definition (cf. [**Daf10**]) the system (3.34) is hyperbolic since the Jacobian matrix $DF(U)$

$$DF(U) = \begin{pmatrix} 0 & 0 & 1 \\ \frac{-m_1 m_2}{\rho^2} & \frac{m_2}{\rho} & \frac{m_1}{\rho} \\ \frac{-{m_2}^2}{\rho^2} + p'(\rho) & 0 & \frac{2m_2}{\rho} \end{pmatrix}$$

has real eigenvalues

$$\lambda_1 = \frac{m_2}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2 = \frac{m_2}{\rho}, \qquad \lambda_3 = \frac{m_2}{\rho} + \sqrt{p'(\rho)} \tag{3.38}$$

and 3 linearly independent eigenvectors

(3.39)
$$R_1 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} - \sqrt{p'(\rho)} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} + \sqrt{p'(\rho)} \end{pmatrix}.$$

The eigenvalue λ_i of DF, $i = 1, 2, 3$, is called the *i-characteristic speed* of the system (3.34). On the part of the state space of our interest, with $\rho > 0$, the system (3.34) is indeed strictly hyperbolic. Finally, one can easily verify that the functions

(3.40)
$$w_3 = \frac{m_2}{\rho} + \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau, \qquad w_2 = \frac{m_1}{\rho}, \qquad w_1 = \frac{m_2}{\rho} - \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau$$

are, respectively, (1- and 2-), (1- and 3-), (2- and 3-) Riemann invariants of the system (3.34) (for the relevant definitions see [**Daf10**]).

In order to characterize rarefaction waves of the reduced system (3.34), we can refer to Theorem 7.6.6 from [**Daf10**]: every i- Riemann invariant is constant along any i- rarefaction wave curve of the system (3.34) and conversely the i- rarefaction wave curve, through a state $(\overline{\rho}, \overline{m})$ of genuine nonlinearity of the i- characteristic family, is determined implicitly by the system of equations $w_i(\rho, m) = w_i(\overline{\rho}, \overline{m})$ for every i- Riemann invariant w_i. As an application of this Theorem, we obtain that the (ρ_R, m_R) lies on the 1- rarefaction wave through (ρ_L, m_L). Indeed, the 1- rarefaction wave of the system (3.34) through the point (ρ_L, m_L) is determined in terms of the Riemann invariants w_3 and w_2 by the equations

$$m_1 = -\frac{\rho}{\rho_L}, \qquad m_2 = \rho \int_\rho^{\rho_L} \frac{\sqrt{p'(\tau)}}{\tau} d\tau, \tag{3.41}$$

with $\rho < \rho_L$. In the case of pressure law $p(\rho) = \rho^2$, the equations (3.41) read as

$$m_1 = -\frac{\rho}{\rho_L}, \qquad m_2 = 2\sqrt{2}\rho \left(\sqrt{\rho_L} - \sqrt{\rho}\right). \tag{3.42}$$

Clearly, the constant state (ρ_R, m_R), as defined by (3.35)–(3.36), satisfies equations (3.42). Since, according to Theorem 7.6.5 in [**Daf10**], there exists a unique 1-rarefaction wave through (ρ_L, m_L), we have shown the existence of our desired self-similar locally Lipschitz solution.

Observe that, by construction, $\rho_+ = \rho_L \geq \rho \geq \rho_R = \rho_- > 0$, thereby showing (i). The claim (iv) follows easily because there exists a constant $C > 0$ such that, for every positive time t, the pair (ρ, m) takes the constant value (ρ_R, m_R) for $x_2 \geq Ct$ and (ρ_L, m_L) for $x_2 \leq -Ct$. ☐

We next show the existence of a solution of the algebraic constraints of Proposition 3.3.1 such that in addition $(\rho_\pm, v_\pm)$ satisfy the identities of Lemma 3.4.1.

Lemma 3.4.2. *Let $p(\rho) = \rho^2$. There exist $\rho_\pm, v_\pm$ satisfying the assumptions of Lemma 3.4.1 and $\rho_1, C_1, v_1, u_1, \nu_\pm$ satisfying the algebraic identities and inequalities* (3.22)-(3.31).

Proof. Taking into account that $p(\rho) = \rho^2$ and, therefore, $\varepsilon(\rho) = \rho$, we substitute the identities of Lemma 3.4.1 into the unknowns of Proposition 3.3.1 and reduce (3.25)-(3.27) to

$$\nu_+(\rho_1 - \rho_+) = \rho_1 \beta \tag{3.43}$$

$$\nu_+(\rho_1 \alpha + 1) = \rho_1 \delta \tag{3.44}$$

$$\nu_+ \rho_1 \beta = -\rho_1 \gamma + \rho_1^2 - \rho_+^2 + \rho_1 \frac{C_1}{2}\,. \tag{3.45}$$

Similarly, we reduce (3.22)-(3.24) to

$$\nu_-(\rho_- - \rho_1) = 2\sqrt{2}\rho_-(\sqrt{\rho_+} - \sqrt{\rho_-}) - \rho_1 \beta \tag{3.46}$$

$$\nu_-(-\frac{\rho_-}{\rho_+} - \rho_1 \alpha) = -2\sqrt{2}\frac{\rho_-}{\rho_+}(\sqrt{\rho_+} - \sqrt{\rho_-}) - \rho_1 \delta \tag{3.47}$$

$$\begin{aligned}\nu_-(2\sqrt{2}\rho_-(\sqrt{\rho_+} - \sqrt{\rho_-}) - \rho_1 \beta)\\ = 8\rho_-(\sqrt{\rho_+} - \sqrt{\rho_-})^2 + \rho_1 \gamma + \rho_-^2 - \rho_1^2 - \rho_1 \frac{C_1}{2}\,.\end{aligned} \tag{3.48}$$

The identities of Lemma 3.4.1 do not influence the form of (3.28)-(3.29). Instead, plugging them into (3.30)-(3.31) the latter are reduced to:

$$\begin{aligned}
(3.49)\qquad & \nu_-\left(\rho_-^2-\rho_1^2+\frac{\rho_-}{2\rho_+^2}+4\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2-\frac{C_1\rho_1}{2}\right)\\
&\le\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})\left(4\rho_-+\frac{1}{\rho_+^2}+8\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2\right)\\
&\quad-2\rho_1^2\beta-\frac{\beta C_1\rho_1}{2}\\
(3.50)\qquad & \nu_+\left(\rho_1^2-\rho_+^2+\frac{C_1\rho_1}{2}-\frac{1}{2\rho_+}\right)\le 2\rho_1^2\beta+\frac{C_1\rho_1\beta}{2}\,.
\end{aligned}$$

We next make the choice $\nu_+=\beta=\delta=0$ and hence (3.43), (3.44) and (3.50) are automatically satisfied. The remaining constraints above become then:

$$\begin{aligned}
(3.51)\qquad & 0=-\rho_1\gamma+\rho_1^2-\rho_+^2+\rho_1\frac{C_1}{2}\\
(3.52)\qquad & \nu_-(\rho_--\rho_1)=2\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})\\
(3.53)\qquad & \nu_-\left(-\frac{\rho_-}{\rho_+}-\rho_1\alpha\right)=-2\sqrt{2}\frac{\rho_-}{\rho_+}(\sqrt{\rho_+}-\sqrt{\rho_-})\\
(3.54)\qquad & \\
&\nu_-(2\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-}))=8\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2+\rho_1\gamma+\rho_-^2-\rho_1^2-\rho_1\frac{C_1}{2}
\end{aligned}$$

and

$$\begin{aligned}
(3.55)\qquad & \nu_-\left(\rho_-^2-\rho_1^2+\frac{\rho_-}{2\rho_+^2}+4\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2-\frac{C_1\rho_1}{2}\right)\\
&\le\sqrt{2}\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})\left(4\rho_-+\frac{1}{\rho_+^2}+8\rho_-(\sqrt{\rho_+}-\sqrt{\rho_-})^2\right)\,.
\end{aligned}$$

Moreover, the inequalities (3.28) and (3.29) become

$$\begin{aligned}
(3.56)\qquad & \alpha^2<C_1\\
(3.57)\qquad & 0<\left(\frac{C_1}{2}-\alpha^2+\gamma\right)\left(\frac{C_1}{2}-\gamma\right)\,.
\end{aligned}$$

Summarizing we are looking for real numbers $\nu_-<0, 0<\rho_-<\rho_+, \rho_1, \alpha, \gamma$ and C_1 satisfying the set of identities and inequalities (3.51)-(3.57).

We next choose $\rho_- = 1 < 4 = \rho_+$ and simplify further (3.51)-(3.55) as

$$\frac{C_1\rho_1}{2} + \rho_1^2 - \rho_1\gamma - 16 = 0 \tag{3.58}$$

$$\nu_-(1-\rho_1) = 2\sqrt{2} \tag{3.59}$$

$$\nu_-\left(\frac{1}{4} + \alpha\rho_1\right) = \frac{\sqrt{2}}{2} \tag{3.60}$$

$$9 + \rho_1\gamma - \rho_1^2 - \frac{C_1\rho_1}{2} = 2\sqrt{2}\nu_- \tag{3.61}$$

$$\nu_-\left(5 + \frac{1}{32} - \rho_1^2 - \frac{C_1\rho_1}{2}\right) \le \sqrt{2}\left(12 + \frac{1}{16}\right). \tag{3.62}$$

We now observe that (3.59) and (3.60) imply $\alpha = -\frac{1}{4}$ and (3.58)-(3.61) imply $\nu_- = -\frac{7}{2\sqrt{2}}$. Therefore our constraints further simplify to look for ρ_1, γ, C_1 such that

$$\frac{1}{16} < C_1 \tag{3.63}$$

$$0 < \left(\frac{C_1}{2} - \frac{1}{16} + \gamma\right)\left(\frac{C_1}{2} - \gamma\right) \tag{3.64}$$

$$0 = \frac{C_1\rho_1}{2} + \rho_1^2 - \rho_1\gamma - 16 \tag{3.65}$$

$$8 = -7(1-\rho_1) \tag{3.66}$$

$$48 + \frac{1}{4} \ge -7\left(5 + \frac{1}{32} - \rho_1^2 - \frac{C_1\rho_1}{2}\right). \tag{3.67}$$

From (3.66) we derive $\rho_1 = \frac{15}{7}$ and inserting this into (3.65) we infer $\frac{C_1}{2} - \gamma = \frac{559}{105}$. In turn this last identity reduces (3.64) to the inequality

$$C_1 > \frac{1}{16} + \frac{559}{105}. \tag{3.68}$$

The remaining constraints (3.65) and (3.67) simplify to:

$$\frac{C_1}{2} - \gamma = \frac{559}{105} \tag{3.69}$$

$$48 + \frac{1}{4} + 35 + \frac{7}{32} - \frac{225}{7} \ge \frac{15C_1}{2}. \tag{3.70}$$

We therefore see that γ can be read off C_1 through (3.69) and hence the existence of the desired solution is equivalent to the inequality

$$\frac{15}{2}\left(\frac{1}{16} + \frac{559}{105}\right) < 48 + \frac{1}{4} + 35 + \frac{7}{32} - \frac{225}{7},$$

which can be trivially checked. □

Theorem 3.1.1 and Corollary 3.1.2 follow now easily.

Proof. [Proofs of Theorem 3.1.1 and Corollary 3.1.2] Let $p(\rho) = \rho^2$ and consider the $\rho_\pm, v_\pm$ given by Lemma 3.4.2. Applying Propositions 3.3.1 and 3.2.6 we know that there are infinitely many admissible solutions of (3.1)-(3.3) as claimed in the Theorem.

Let now (ρ_f, v_f) be any such solution and let (ρ_b, v_b) be the locally Lipschitz solutions of (3.1) given by Lemma 3.4.1. It is straightforward to check that, if we define

$$(\rho, v)(x,t) := \begin{cases} (\rho_f, v_f)(x,t) & \text{if } t \geq 0 \\ \\ (\rho_b, v_b)(x,t) & \text{if } t \leq 0\,, \end{cases} \tag{3.71}$$

then the pair (ρ, v) is a bounded admissible solution of (3.1) on the entire space-time $\mathbb{R}^2 \times \mathbb{R}$ with density bounded away from 0. Moreover $(\rho(\cdot,t), v(\cdot,t))$ is a bounded Lipschitz function for every $t < 0$. In particular we can define $(\tilde\rho, \tilde v)(x,t) = (\rho, v)(x, t-1)$ and observe that, no matter which of the infinitely many solutions (ρ_f, v_f) given by Theorem 3.1.1 we choose, the corresponding $(\tilde\rho, \tilde v)$ defined above is an admissible solution as in Corollary 3.1.2 for the bounded and Lipschitz initial data $(\rho^0, v^0) = (\rho_b, v_b)(\cdot, -1)$. □

3.5. Second method: further Riemann data for different pressures

In this section we describe a second method to produce solutions to the algebraic set of equations and inequalities of Proposition 3.3.1. Unlike the method given in the previous section, we do not know whether this one produces Riemann data generated by a compression wave. Moreover we do not fix the pressure law but we exploit it as an extra degree of freedom. On the other hand the reader can easily check that the method below gives a rather large set of solutions (i.e. open) compared to the one of Lemma 3.4.2 (where we do not know whether one can perturb the choice $\nu_+ = 0$.

LEMMA 3.5.1. *Set* $v_\pm = (\pm 1, 0)$. *Then there exist* $\nu_\pm$, $\rho_\pm$, ρ_1, $\alpha, \beta, \gamma, \delta$, C_1 *and a smooth pressure* p *with* $p' > 0$ *for which the algebraic identities and inequalities* (3.22)-(3.31) *are satisfied.*

3.5.1. Part I of the proof of Lemma 3.5.1: reduction of the admissibility conditions. We rewrite the conditions (3.22)-(3.27)

$$\nu_-(\rho_1 - \rho_-) = \rho_1\beta \tag{3.72}$$

$$\nu_-(\rho_- + \rho_1\alpha) = \rho_1\delta \tag{3.73}$$

$$\rho_1\frac{C_1}{2} - \rho_1\gamma + p(\rho_1) - p(\rho_-) = \nu_-\rho_1\beta \tag{3.74}$$

$$\nu_+(\rho_1 - \rho_+) = \rho_1\beta \tag{3.75}$$

$$\nu_+(\rho_1\alpha - \rho_+) = \rho_1\delta \tag{3.76}$$

$$\rho_1\frac{C_1}{2} - \rho_1\gamma + p(\rho_1) - p(\rho_+) = \nu_+\rho_1\beta\,. \tag{3.77}$$

The conditions (3.28) and (3.29) are not affected by our choice. The conditions (3.30) and (3.31) become

$$\nu_-\left(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1) + \frac{\rho_-}{2} - \rho_1\frac{C_1}{2}\right) + \beta\left(\rho_1\varepsilon(\rho_1) + p(\rho_1) + \rho_1\frac{C_1}{2}\right) \le 0 \tag{3.78}$$

$$\nu_+\left(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+) + \rho_1\frac{C_1}{2} - \frac{\rho_+}{2}\right) - \beta\left(\rho_1\varepsilon(\rho_1) + p(\rho_1) + \rho_1\frac{C_1}{2}\right) \le 0\,. \tag{3.79}$$

Plugging (3.72) and (3.75) into, respectively, (3.78) and (3.79) we achieve

$$\nu_-\left(\rho_-\varepsilon(\rho_-) - \rho_1\varepsilon(\rho_1) - \rho_1\frac{C_1 - 1}{2}\right) + \beta\left(\rho_1\varepsilon(\rho_1) + p(\rho_1) + \rho_1\frac{C_1 - 1}{2}\right) \le 0 \tag{3.80}$$

$$\nu_+\left(\rho_1\varepsilon(\rho_1) - \rho_+\varepsilon(\rho_+) + \rho_1\frac{C_1 - 1}{2}\right) - \beta\left(\rho_1\varepsilon(\rho_1) + p(\rho_1) + \rho_1\frac{C_1 - 1}{2}\right) \le 0\,. \tag{3.81}$$

We next rely on the following

LEMMA 3.5.2. *Let us suppose that*

$$\nu_- < 0 < \nu_+\,, \tag{3.82}$$

$$\rho_- < \rho_+\,. \tag{3.83}$$

Then, there exist pressure functions $p \in C^\infty([0, +\infty[)$ *with* $p' > 0$ *on* $]0, +\infty[$ *such that the admissibility conditions* (3.80)-(3.81) *for a* *subsolution are implied by the following system of inequalities:*

$$(p(\rho_+) - p(\rho_1))(\rho_+ - \rho_1) > \frac{C_1 - 1}{2}\rho_+\rho_1 \tag{3.84}$$

$$(p(\rho_1) - p(\rho_-))(\rho_1 - \rho_-) > \frac{C_1 - 1}{2}\rho_-\rho_1. \tag{3.85}$$

Proof. First, let us define $g(\rho) := \rho\varepsilon(\rho)$. In view of the relation $p(\rho) = \rho^2\varepsilon'(\rho)$, we obtain

$$g'(\rho) = \varepsilon(\rho) + \frac{p(\rho)}{\rho}.$$

Thus, owing respectively to (3.72) and (3.75), we can rewrite (3.80) and (3.81) as follows:

$$\nu_-(g(\rho_-) - g(\rho_1)) + \nu_-(\rho_1 - \rho_-)g'(\rho_1) - \nu_-\rho_-\frac{C_1 - 1}{2} \leq 0. \tag{3.86}$$

$$\nu_+(g(\rho_1) - g(\rho_+)) + \nu_+(\rho_+ - \rho_1)g'(\rho_1) + \nu_+\rho_+\frac{C_1 - 1}{2} \leq 0 \tag{3.87}$$

From the hypothesis (3.82) we can further reduce (3.86)-(3.87) to

$$-(g(\rho_1) - g(\rho_-)) + (\rho_1 - \rho_-)g'(\rho_1) \geq \frac{C_1 - 1}{2}\rho_-. \tag{3.88}$$

$$(g(\rho_+) - g(\rho_1)) - (\rho_+ - \rho_1)g'(\rho_1) \geq \frac{C_1 - 1}{2}\rho_+ \tag{3.89}$$

Moreover, we observe from (3.72)-(3.75) that

$$\nu_+(\rho_+ - \rho_1) = -\nu_-(\rho_1 - \rho_-).$$

Hence, in view of (3.82)-(3.83), we must have

$$\rho_- < \rho_1 < \rho_+\,. \tag{3.90}$$

Let us note that

$$(g(\sigma) - g(s)) - (\sigma - s)g'(s) = \int_s^\sigma \int_s^\tau g''(r)\,dr\,d\tau$$

for every $s < \sigma$. On the other hand, by simple algebra, we can compute $g''(r) = p'(r)/r$. Hence, the following equalities hold for every $s < \sigma$:

$$(g(\sigma) - g(s)) - (\sigma - s)g'(s) = \int_s^\sigma \int_s^\tau \frac{p'(r)}{r}\,dr\,d\tau$$

and

$$(g(s) - g(\sigma)) + (\sigma - s)g'(\sigma) = \int_s^\sigma \int_\tau^\sigma \frac{p'(r)}{r}\,dr\,d\tau.$$

As a consequence, and in view of (3.90), we can rewrite (3.88) and (3.89) equivalently as

$$\int_{\rho_-}^{\rho_1}\int_{\tau}^{\rho_1}\frac{p'(r)}{r}drd\tau \geq \frac{C_1-1}{2}\rho_-. \tag{3.91}$$

$$\int_{\rho_1}^{\rho_+}\int_{\rho_1}^{\tau}\frac{p'(r)}{r}drd\tau \geq \frac{C_1-1}{2}\rho_+, \tag{3.92}$$

Now, we introduce two new variables q_- and q_+ defined by

$$q_- := p(\rho_1) - p(\rho_-)\,,$$
$$q_+ := p(\rho_+) - p(\rho_1)\,.$$

Proving Lemma 3.5.2 is then equivalent to show the existence of a pressure law p satisfying $p(\rho_+) - p(\rho_1) = q_+$, $p(\rho_1) - p(\rho_-) = q_-$ and for which the inequalities (3.91)-(3.92) hold.

First, introducing $f := p'$, we define the set of functions

$$\mathcal{L} := \left\{ f \in C^\infty(]0,\infty[,]0,\infty[) \;:\; \int_{\rho_-}^{\rho_1} f = q_- \text{ and } \int_{\rho_1}^{\rho_+} f = q_+ \right\}$$

and the two functionals defined on $\mathcal{L}$

$$L^+(f) := \int_{\rho_1}^{\rho_+}\int_{\rho_1}^{\tau}\frac{f(r)}{r}drd\tau,$$

$$L^-(f) := \int_{\rho_-}^{\rho_1}\int_{\tau}^{\rho_1}\frac{f(r)}{r}drd\tau.$$

Therefore, sufficient conditions to find a pressure function p with the properties above is that

$$l^+ := \sup_{f\in\mathcal{L}} L^+(f) > \frac{C_1-1}{2}\rho_+$$

and

$$l^- := \sup_{f\in\mathcal{L}} L^-(f) > \frac{C_1-1}{2}\rho_-.$$

Let us generalize the space $\mathcal{L}$ as follows. We introduce

$$\mathcal{M}^+ := \{\text{positive Radon measures } \mu \text{ on } [\rho_1,\rho_+] : \mu([\rho_1,\rho_+]) = q_+\}\,,$$

$$\mathcal{M}^- := \{\text{positive Radon measures } \mu \text{ on } [\rho_-,\rho_1] : \mu([\rho_-,\rho_1]) = q_-\}\,.$$

Consistently, we extend the functionals L^+ and L^- defined on $\mathcal{L}$ to new functionals L_+ and L_- respectively defined on $\mathcal{M}^+$ and on $\mathcal{M}^-$:

$$L_+(\mu) := \int_{\rho_1}^{\rho_+}\int_{\rho_1}^{\tau}\frac{1}{r}d\mu(r)d\tau \qquad\qquad \text{for } \mu\in\mathcal{M}^+,$$

$$L_-(\mu) := \int_{\rho_-}^{\rho_1} \int_{\tau}^{\rho_1} \frac{1}{r} d\mu(r) d\tau \qquad \text{for } \mu \in \mathcal{M}^-.$$

Once introduced

$$m^+ := \max_{\mu \in \mathcal{M}^+} L_+(\mu)$$

and

$$m^- := \max_{\mu \in \mathcal{M}^-} L_-(\mu),$$

it is clear that

$$l^+ \leq m^+ \quad \text{and} \quad l^- \leq m^-.$$

Moreover, let us remark the existence of $m^{\pm}$ (i.e. that the maxima are achieved) due to the compactness of $\mathcal{M}^{\pm}$ with respect to the weak* topology. By a simple Fubini's type argument, we write

$$L_+(\mu) = \int_{\rho_1}^{\rho_+} \frac{\rho_+ - r}{r} \, d\mu(r).$$

Hence, defining the function $h \in C([\rho_1, \rho_+])$ as $h(r) := (\rho_+ - r)/r$ allows us to express the action of the linear functional L_+ as a duality pairing; more precisely we have:

$$L_+(\mu) =< h, \mu > \qquad \text{for } \mu \in \mathcal{M}^+.$$

Analogously, if we define $g \in C([\rho_-, \rho_1])$ as $g(r) := (r - \rho_-)/r$, we can express L_- as a duality pairing as well:

$$L_-(\mu) =< g, \mu > \qquad \text{for } \mu \in \mathcal{M}^-.$$

By standard functional analysis, we know that $m^{\pm}$ must be achieved at the extreme points of $\mathcal{M}^{\pm}$. The extreme points of $\mathcal{M}^{\pm}$ are the single-point measures, i.e. weighted Dirac masses. For $\mathcal{M}^+$ the set of extreme points is then given by $E_+ := \{q_+\delta_\sigma \text{ for } \sigma \in [\rho_1, \rho_+]\}$ while for $\mathcal{M}^-$ the set of extreme points is then given by $E_- := \{q_-\delta_\sigma \text{ for } \sigma \in [\rho_-, \rho_1]\}$ In order to find $m^{\pm}$, it is enough to find the maximum value of $L_{\pm}$ on $E_{\pm}$. Clearly, we obtain

$$m^+ = \max_{\sigma \in [\rho_1, \rho_+]} \left\{ q_+ \frac{\rho_+ - \sigma}{\sigma} \right\} = q_+ \frac{\rho_+ - \rho_1}{\rho_1}$$

and

$$m^- = \max_{\sigma \in [\rho_-, \rho_1]} \left\{ q_- \frac{\sigma - \rho_-}{\sigma} \right\} = q_- \frac{\rho_1 - \rho_-}{\rho_1}.$$

Furthermore, for every $\varepsilon > 0$ there exists a function $f \in \mathcal{L}$ such that

$$L^+(f) > q_+ \frac{\rho_+ - \rho_1}{\rho_1} - \varepsilon$$

and

$$L_-(f) > q_- \frac{\rho_1 - \rho_-}{\rho_1} - \varepsilon.$$

Such a function f is the derivative of the desired pressure function p. $\square$

3.5.2. Part II of the proof of Lemma 3.5.1. We now choose $\rho_1 = 1$. Applying Lemma 3.5.2 we set $q_\pm := \pm(p(\rho_\pm) - p(\rho_1)) = \pm(p(\rho_\pm) - p(1))$ and hence reduce our problem to find real numbers $\rho_\pm, \nu_\pm, q_\pm, \alpha, \beta, \gamma, \delta, C_1$ satisfiyng

$$\nu_- < 0 < \nu_+\,, \quad 0 < \rho_- < 1 < \rho_+\,, \quad q_\pm > 0 \tag{3.93}$$

$$\nu_-(1-\rho_-) = \beta \tag{3.94}$$

$$\nu_-(\rho_- + \alpha) = \delta \tag{3.95}$$

$$\frac{C_1}{2} - \gamma + q_- = \nu_-\beta \tag{3.96}$$

$$\nu_+(1-\rho_+) = \beta \tag{3.97}$$

$$\nu_+(\alpha - \rho_+) = \delta \tag{3.98}$$

$$\frac{C_1}{2} - \gamma - q_+ = \nu_+\beta\,, \tag{3.99}$$

$$q_-(1-\rho_-) \geq \frac{C_1 - 1}{2}\rho_- \tag{3.100}$$

$$q_+(\rho_+ - 1) \geq \frac{C_1 - 1}{2}\rho_+ \tag{3.101}$$

and (3.28)-(3.29).

Next, using (3.93), (3.94) and (3.97) we rewrite (3.100)-(3.101) as

$$-\beta q_- > \frac{C_1 - 1}{2}(-\nu_-\rho_-) \tag{3.102}$$

$$-\beta q_+ > \frac{C_1 - 1}{2}\nu_+\rho_+\,. \tag{3.103}$$

In order to simplify our computations we then introduce the new variables

$$\overline{\beta} = -\beta\,,\ \overline{\delta} = -\delta\,,\ \overline{C} = \frac{C_1}{2}\,,\ \nu^- = -\nu_-\,, \tag{3.104}$$

$$r_+ = \rho_+\nu_+ \text{ and } r_- = \rho_-\nu^- = -\rho_-\nu_-\,.$$

Our conditions become therefore

$$q_\pm, r_\pm, \nu_+, \nu^- > 0 \tag{3.105}$$

$$\nu^- - r_- = \overline{\beta} \tag{3.106}$$
$$r_+ - \nu_+ = \overline{\beta} \tag{3.107}$$
$$r_- + \alpha\nu^- = \overline{\delta} \tag{3.108}$$
$$r_+ - \alpha\nu_+ = \overline{\delta} \tag{3.109}$$
$$\overline{C} - \gamma + q_- = \nu^-\overline{\beta} \tag{3.110}$$
$$\overline{C} - \gamma - q_+ = -\nu_+\overline{\beta}\,, \tag{3.111}$$

$$\overline{\beta}q_- > \left(\overline{C} - \frac{1}{2}\right) r_- \tag{3.112}$$
$$\overline{\beta}q_+ > \left(\overline{C} - \frac{1}{2}\right) r_+ \tag{3.113}$$

and finally (3.28)-(3.29) become

$$\alpha^2 + \overline{\beta}^2 < 2\overline{C} \tag{3.114}$$
$$(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma) - (\overline{\delta} - \alpha\overline{\beta})^2 > 0\,. \tag{3.115}$$

We assume $\alpha^2 \neq 1$ and solve for ν_-, ν^+ and $r_\pm$ in (3.106)-(3.109) to achieve

$$\nu^- = \frac{\overline{\delta} + \overline{\beta}}{1+\alpha}\,, \quad \nu_+ = \frac{\overline{\delta} - \overline{\beta}}{1-\alpha} \quad \text{and} \quad r_\pm = \frac{\overline{\delta} - \alpha\overline{\beta}}{1 \mp \alpha}\,. \tag{3.116}$$

Observe that

$$r_+ r_- = \frac{(\overline{\delta} - \alpha\overline{\beta})^2}{1-\alpha^2}\,.$$

Hence, if we assume $\alpha^2 < 1$ and $\overline{\delta} > \overline{\beta} > 0$, we see that the $\nu^+, \nu_-, r_\pm$ as defined in the formulas (3.116) fulfill the inequalities in (3.105). Hence, inserting (3.116) we look for solutions of the set of identities and inequalities

$$\alpha^2 < 1\,,\ \overline{\delta} > \overline{\beta} > 0\,,\ q_\pm > 0 \tag{3.117}$$

$$\overline{C} - \gamma + q_- = \frac{\overline{\delta} + \overline{\beta}}{1+\alpha}\overline{\beta} \tag{3.118}$$

$$\overline{C} - \gamma - q_+ = -\frac{\overline{\delta} - \overline{\beta}}{1-\alpha}\overline{\beta} \tag{3.119}$$

$$\overline{\beta} q_- > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1+\alpha} \tag{3.120}$$

$$\overline{\beta} q_+ > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1-\alpha} \tag{3.121}$$

combined with (3.114) and (3.115). Observe that, if we assume in addition that $\bar{C} > \frac{1}{2}$, then $\alpha^2 < 1$, $\overline{\delta} > \overline{\beta} > 0$ and (3.120)-(3.121) imply the positivity of $q_\pm$. We can therefore solve for $q_\pm$ the equations (3.118)-(3.119) and insert the corresponding values in the remaining inequalities. Summarizing, we are looking for $\alpha, \overline{\beta}, \gamma, \overline{\delta}, \overline{C}$ fulfilling the following inequalities

$$\alpha^2 < 1\,,\ \overline{\delta} > \overline{\beta} > 0\,,\ \overline{C} > \frac{1}{2} \tag{3.122}$$

$$\overline{\beta}\left[\overline{\beta}\frac{\overline{\delta} + \overline{\beta}}{1+\alpha} - \overline{C} + \gamma\right] > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1+\alpha} \tag{3.123}$$

$$\overline{\beta}\left[\overline{\beta}\frac{\overline{\delta} - \overline{\beta}}{1-\alpha} + \overline{C} - \gamma\right] > \left(\overline{C} - \frac{1}{2}\right)\frac{\overline{\delta} - \alpha\overline{\beta}}{1-\alpha} \tag{3.124}$$

$$\alpha^2 + \overline{\beta}^2 < 2\overline{C} \tag{3.125}$$

$$(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma) - (\overline{\delta} - \alpha\overline{\beta})^2 > 0\,. \tag{3.126}$$

We next introduce the variable $\lambda = \overline{\delta} - \alpha\overline{\beta}$ and rewrite our inequalities as

$$\alpha^2 < 1\,,\ \lambda > (1-\alpha)\overline{\beta} > 0\,,\ \overline{C} > \frac{1}{2} \tag{3.127}$$

$$\overline{\beta}(1+\alpha)(\overline{\beta}^2 - \overline{C} + \gamma) > \left(\overline{C} - \overline{\beta}^2 - \frac{1}{2}\right)\lambda \tag{3.128}$$

$$\overline{\beta}(1-\alpha)(-\overline{\beta}^2 + \overline{C} - \gamma) > \left(\overline{C} - \overline{\beta}^2 - \frac{1}{2}\right)\lambda \tag{3.129}$$

$$\alpha^2 + \overline{\beta}^2 < 2\overline{C} \tag{3.130}$$

$$(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma) > \lambda^2\,. \tag{3.131}$$

Observe that, if we require $\alpha, \overline{\beta}, \gamma$ and $\overline{C}$ to satisfy the following inequalities

$$\alpha^2 < 1\,,\ \overline{C} > \frac{1}{2} \tag{3.132}$$

$$\overline{C} - \alpha^2 + \gamma > 0 \tag{3.133}$$

$$\overline{C} - \overline{\beta}^2 - \gamma > 0 \tag{3.134}$$

$$\overline{\beta}^2 + \frac{1}{2} - \overline{C} > 0 \tag{3.135}$$

$$\sqrt{(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma)} > (1-\alpha)\overline{\beta} > 0 \tag{3.136}$$

$$\left(\overline{\beta}^2 + \frac{1}{2} - \overline{C}\right)\sqrt{\overline{C} - \alpha^2 + \gamma} > \overline{\beta}(1+\alpha)\sqrt{\overline{C} - \overline{\beta}^2 - \gamma} \tag{3.137}$$

then setting

$$\lambda := \sqrt{(\overline{C} - \alpha^2 + \gamma)(\overline{C} - \overline{\beta}^2 - \gamma)} - \eta\,,$$

the inequalities (3.127)-(3.131) are satisfied whenever η is a sufficiently small positive number.

Observe next that (3.135) is surely satisfied if the remaining inequalities are and hence we can drop it. Moreover, if $\overline{\beta}, \gamma$ and $\overline{C}$ satisfy

$$\overline{\beta} > 0\,,\ \overline{C} > \frac{1}{2} \tag{3.138}$$

$$\overline{C} - \overline{\beta}^2 - \gamma > 0 \tag{3.139}$$

$$\overline{C} - 1 + \gamma > 0 \tag{3.140}$$

$$\left(\overline{\beta}^2 + \frac{1}{2} - \overline{C}\right)\sqrt{\overline{C} - 1 + \gamma} > 2\overline{\beta}\sqrt{\overline{C} - \overline{\beta}^2 - \gamma} \tag{3.141}$$

then setting $\alpha = 1 - \vartheta$, the inequalities (3.132)-(3.137) hold provided $\vartheta > 0$ is chosen small enough.

Finally, choosing $\overline{C} = \frac{4}{5}\overline{\beta}^2$, $\gamma = -\frac{2}{5}\overline{\beta}^2$ and imposing $\overline{\beta} > \sqrt{\frac{5}{2}}$ we see that (3.138), (3.139) and (3.140) are automatically satisfied. Whereas (3.141) is equivalent to

$$\left(\frac{\overline{\beta}^2}{5} + \frac{1}{2}\right)\sqrt{\frac{2\overline{\beta}^2}{5} - 1} > \frac{2\overline{\beta}^2}{\sqrt{5}}\,.$$

However the latter inequality is surely satisfied for $\overline{\beta}$ large enough.

3.6. Weak solutions to the incompressible Euler equations with vortex sheet initial data

Recently Székelyhidi constructed infinitely many admissible weak solutions to the incompressible Euler equations in two space dimensions with initial data given by the classical vortex sheet. He considered the Cauchy problem for the incompressible Euler equations (see Section 1.3.2),

$$\begin{cases} \mathrm{div}_x v \ = \ 0 \\ \partial_t v + \mathrm{div}_x\,(v \otimes v) + \nabla_x p \ = \ 0 \\ v(\cdot, 0) \ = \ v^0 \end{cases} . \tag{3.142}$$

where the unknowns v and p are the velocity vector and the pressure. His construction is based on the "convex integration" method introduced recently in [**DLS10**]. His result inspired us and in particular suggested that a similar approach could be of interest also for the compressible Euler system thus leading to Theorem 3.0.1.

The starting point of Székelyhidi's construction lies in the approach of [**DLS09**]-[**DLS10**] towards the construction of weak solutions to the incompressible Euler equations (3.142) in order to recover the celebrated non–uniqueness results of Scheffer [**Sch93**] and Shnirelman [**Shn97**] (see Section 1.3.2.1). The method in [**DLS09**]-[**DLS10**] is a revisitation of convex integration and Baire category arguments. In particular, in [**DLS10**] the strategy behind the construction of "admissible" weak solutions to the initial value problem was based on the notion of subsolution (we refer to Section 1.3.2 in the Introduction of the thesis for the relevant definition). Thanks to this strategy, in [**DLS10**] it was shown that admissibility by itself does not imply uniqueness for the incompressible Euler system (3.142). In other words there exist initial data v^0, for which there exist infinitely many distinct admissible weak solutions of the incompressible Euler equations (3.142). Such initial data are called *wild* initial data in [**DLS10**]. In particular, one can show the existence of infinitely many weak solutions satisfying the Duchon-Robert admissibility condition (see section 1.3.2)

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \leq 0 \tag{3.143}$$

in the sense of distributions, i.e. such that

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|v|^2}{2}\partial_t \phi + \left(\frac{|v|^2}{2} + p\right) v \cdot \nabla \phi \geq 0 \tag{3.144}$$

for every nonnegative $\phi \in C_c^\infty(\mathbb{R}^n \times]0, \infty[)$.

Obviously, wild initial data have to possess a certain amount of irregularity. This follows from the weak- strong uniqueness and classical local existence results. From the construction of wild initial data in [**DLS10**] it was not clear how bad this irregularity needs to be. In [**Sz11**] Székelyhidi showed that the classical vortex-sheet with a flat interface is a wild initial data in two space dimensions. More precisely, consider the following solenoidal vector field in $\mathbb{R}^2$

$$v^0(x) := \begin{cases} v^+ := (1, 0) & \text{if } x_2 > 0, \\ v^- := (-1, 0) & \text{if } x_2 < 0, \end{cases} \tag{3.145}$$

then the following theorem holds:

Theorem 3.6.1 (The vortex sheet is wild). *For v^0 as in* (3.145) *there are infinitely many weak solutions of* (3.142) *on $\mathbb{R}^2 \times [0, \infty[$ which satisfy the admissibility condition* (3.143) *in the sense of distributions.*

As already explained in section 1.3.2, Theorem 3.6.1 is proved in [**Sz11**] using an adapted version of Proposition 1.3.5 which is here reformulated in Lemma 3.2.7: hence the proof essentially consisted in finding a suitable subsolution for the incompressible Euler system which takes the right initial values. In [**Sz11**] the construction of such a subsolution follows an idea introduced in [**Szé11**] for the incompressible porous media equation. Here we will show that the existence of such a subsolution can be achieved also in a more direct way (cf. [**Shn97**]) which inspired us for the compressible case. In the next section we will show how Lemma 3.2.7 implies Theorem 3.6.1.

3.6.1. Direct proof of Theorem 3.6.1. The aim of this section is to apply Lemma 3.2.7 in order to prove Theorem 3.6.1. Our starting point is to find a triple $(\overline{v}, \overline{u}, \overline{q})$ satisfying

$$\begin{cases} \operatorname{div}_x \overline{v} \ = \ 0 \\ \partial_t \overline{v} + \operatorname{div}_x \overline{u} + \nabla_x \overline{q} \ = \ 0 \end{cases} \tag{3.146}$$

in the sense of distributions. We will denote the space variable as $x = (x_1, x_2) \in \mathbb{R}^2$. To this aim, we consider potential subsolutions of the following form:

$$\begin{aligned} (\overline{v}, \overline{u}, \overline{q}) = (v^-, u^-, q^-)&\mathbf{1}_{R^-} \\ + (\hat{v}, \hat{u}, \hat{q})&\mathbf{1}_R \\ + (v^+, u^+, q^+)&\mathbf{1}_{R^+}, \end{aligned} \tag{3.147}$$

with

$$\begin{aligned} R^- &:= \{(x,t) :\ t > 0 \text{ and } x_2 < \nu_1 t\}, \\ R &:= \{(x,t) :\ t > 0 \text{ and } \nu_1 t < x_2 < \nu_2 t\}, \\ R^+ &:= \{(x,t) :\ t > 0 \text{ and } x_2 > \nu_2 t\} \end{aligned}$$

and

$$\hat{v} = (\alpha, 0), \tag{3.148}$$

$$u^- = u^+ = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{3.149}$$

$$q^- = q^+ = \frac{1}{2}, \tag{3.150}$$

$$\hat{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \tag{3.151}$$

for some constants $\nu_1 < 0 < \nu_2$, α, β, γ and $\hat{q}$. Of course such a choice is reminiscent of fan subsolutions for compressible Euler. Inside each of the three regions R^-, R and R^+ the equations defining a subsolution are trivially satisfied; hence they need to be imposed only along *fronts* which do not depend on x_1. Since the divergence free condition is trivially satisfied for our choice of $\hat{v}$, the system (3.146) simply reads as

$$\nu_2(\alpha - 1) = \gamma, \tag{3.152a}$$

$$\beta = \hat{q}, \tag{3.152b}$$

$$\nu_1(\alpha + 1) = \gamma. \tag{3.152c}$$

Finally, if we choose $\tilde{v} = \hat{v}$, $\tilde{u} = \hat{u}$, $C = 1$ and $\Omega = R$ in the assumptions of Lemma 3.2.7, then the requirement of the Lemma amounts to the condition

$$\begin{pmatrix} \alpha^2 - \beta & -\gamma \\ -\gamma & \beta \end{pmatrix} < \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \tag{3.153}$$

which is equivalent to the following couple of inequalities

$$\alpha^2 < 1, \tag{3.154a}$$

$$\frac{1}{4} - \frac{1}{2}\alpha^2 + \beta\alpha^2 - \beta^2 - \gamma^2 > 0. \tag{3.154b}$$

With the choice $\alpha = 0$, $\nu_1 = -\nu_2 = \gamma < 0$ and $-\gamma = \beta = \hat{q} = \frac{1}{4}$, all the conditions (3.152a)-(3.154b) are satisfied. In particular we can apply Lemma 3.2.7 and find infinitely many functions $(\underline{v}, \underline{u})$ satisfying the property (ii) in the Lemma on $\mathbb{R}^2\times]0,\infty[$ and such that the following holds:

$$(\hat{v} + \underline{v}) \otimes (\hat{v} + \underline{v}) - (\hat{u} + \underline{u}) = \frac{1}{2}\mathrm{Id} \text{ a.e. in } R, \tag{3.155}$$

$$(\underline{v}, \underline{u}) = 0 \text{ a.e. on } R^c. \tag{3.156}$$

Next, we define v and u in a highly non-unique way as follows:

$$v := \overline{v} + \underline{v},$$

$$u := \overline{u} + \underline{u}.$$

Thanks to (3.146), (3.155)-(3.156), the infinitely many functions v so constructed are weak solutions of (3.142) on $\mathbb{R}^2\times]0,\infty[$ with initial data v^0 and with pressure $p = \overline{q} - \frac{1}{2}$. Indeed, owing to (3.146) and (3.155)-(3.156), we have in the sense of distributions:

$$\begin{aligned}
&\partial_t v + \operatorname{div}_x(v \otimes v) + \nabla_x p = \\
&\partial_t \overline{v} + \partial_t \underline{v} + \operatorname{div}_x \left[(\overline{v} + \underline{v}) \otimes (\overline{v} + \underline{v})\right] + \nabla_x \left(\overline{q} - \frac{1}{2}\right) = \\
&\partial_t \overline{v} + \partial_t \underline{v} + \operatorname{div}_x \left(\overline{u} + \underline{u} + \frac{1}{2} Id\right) + \nabla_x \left(\overline{q} - \frac{1}{2}\right) = \\
&(\partial_t \overline{v} + \operatorname{div}_x \overline{u} + \nabla_x \overline{q}) + (\partial_t \underline{v} + \operatorname{div}_x \underline{u}) = 0.
\end{aligned}$$

Similarly one can prove that v is weakly divergence–free. Observe that so far we have shown that (3.142) holds in the sense of distributions whenever the corresponding test functions are supported in $\mathbb{R}^2 \times]0, \infty[$. However observe that, since as $\tau \downarrow 0$ the Lebesgue measure of $R \cap \{t = \tau\}$ converges to 0, the map $v(\tau, \cdot)$ converge to the maps v^0 of (3.145) strongly in L^1_{loc}. This easily implies (3.142) in its full generality: indeed one could argue as in the proof of Proposition 3.2.6.

Finally, also the admissibility condition (3.144) is satisfied by the so constructed infinitely many weak solutions of the incompressible Euler equations. More precisely, since the modulus of v is almost everywhere constant ($|v| = |\overline{v} + \underline{v}| = 1$), v is weakly divergence–free and so is $\overline{v}$, we have in the sense of distributions:

$$\begin{aligned}
&\partial_t \frac{|v|^2}{2} + \operatorname{div}_x \left(\left(\frac{|v|^2}{2} + p\right) v\right) = \\
&\frac{|v|^2}{2} \operatorname{div}_x v + \operatorname{div}_x(pv) = \\
&\operatorname{div}_x(\overline{q} v) - \frac{1}{2} \operatorname{div}_x v = \\
&\operatorname{div}_x(\overline{q}\, \overline{v}) + \operatorname{div}_x(\overline{q} \underline{v}) = \\
&\nabla_x \overline{q} \cdot \overline{v} + \operatorname{div}_x(\overline{q} \underline{v}) = 0,
\end{aligned}$$

where the last equality is motivated by the facts that $\partial_{x_1} \overline{q} = 0$ and $\overline{v}_2 = 0$ and that $\overline{q}$ is constant on the support of $\underline{v}$ which on the other hand is divergence–free. This concludes the proof of Theorem 3.6.1.

CHAPTER 4

Classical solution of Riemann problems for isentropic Euler

This chapter is a complement to Chapter 3. Here we restrict our attention to the 1-*dimensional Riemann problem* for the compressible Euler equations with the same choice of Riemann initial data allowing for the non–uniqueness theorems proven in Chapter 3 (see Theorem 3.1.1) (such data indeed depend only on one space variable): we show that such a problem admits unique self-similar solutions. This follows from classical considerations but since we have not been able to find a precise reference, we include the argument for completeness. In particular we will first investigate in Section 4.1 the case of quadratic pressure law and of Riemann data generated by compression waves (as in Theorem 3.1.1) and then show in Section 4.2 uniqueness of self–similar solutions also for the choices of data and pressures carried out in Section 3.5 of Chapter 3. Theorem 3.1.1 of Chapter 3 shows that as soon as the self–similarity assumption runs out, uniqueness is lost.

4.1. Riemann data generated by compression wave

In this section, we consider the Riemann problem for the compressible Euler equations

$$
\begin{cases}
\partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0\\
\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] \;=\; 0\\
\rho(\cdot,0) \;=\; \rho^0\\
v(\cdot,0) \;=\; v^0\,.
\end{cases}
\tag{4.1}
$$

with quadratic pressure law $p(\rho) = \rho^2$. In particular we deal with Riemann data of the form

$$
(\rho^0(x), v^0(x)) := \begin{cases}
(\rho_-, v_-) & \text{if } x_2 < 0\\
\\
(\rho_+, v_+) & \text{if } x_2 > 0,
\end{cases}
\tag{4.2}
$$

which can be generated by compression waves and we aim at proving uniqueness of self–similar solutions to (4.1)–(4.2). This uniqueness result should follow from classical theory but since we have not been able to find a complete reference, we present here the arguments.

In order to achieve our goal we need to refer to Chapter 3. More precisely, we choose $\rho_\pm$ and $v_\pm$ in (4.2) as dictated by Lemma 3.4.1 in Chapter 3 (which therefore include the data allowing for infinitely many admissible solutions forward in time as in Theorem 3.1.1 and Corollary 3.1.2), that is we let $0 < \rho_- < \rho_+$ and

(4.3)
$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, (-\frac{1}{\rho_+}, 2\sqrt{2}(\sqrt{\rho_+} - \sqrt{\rho_-}))) & \text{if } x_2 < 0 \\ \\ (\rho_+, (-\frac{1}{\rho_+}, 0)) & \text{if } x_2 > 0. \end{cases}$$

Then we have the following proposition

PROPOSITION 4.1.1. *Consider* $p(\rho) = \rho^2$ *and any initial data of type* (4.3)*. Then there exists a unique admissible self-similar bounded* BV_{loc} *solution (i.e. of the form* $(\rho, v)(x,t) = (r, w)(\frac{x_2}{t})$*) of* (4.1) *with* ρ *bounded away from* 0*.*

REMARK 4.1.2. *In fact the following proof of Proposition 4.1.1 has a much stronger outcome. In particular the same uniqueness conclusion holds under the following more general assumptions:*

- p *satisfies the usual "hyperbolicity assumption"* $p' > 0$ *and the "genuinely nonlinearity condition"* $2p'(r) + rp''(r) > 0\ \forall r > 0$*;*
- (ρ, v) *is a bounded admissible solution with density bounded away from zero, whereas the* BV *regularity and the self-similarity hypotheses are assumed* only *for* ρ *and the second component of the velocity* v*.*

PROOF. Observe that the initial data for the first component v_1 is the constant $-\frac{1}{\rho_+}$. On the other hand:

- ρ is a bounded function of locally bounded variation;
- The vector field $\bar{v} = (0, v_2)$ is bounded, has locally bounded variation and solves the continuity equation

$$\partial_t \rho + \mathrm{div}_x(\rho \bar{v}) = 0\,; \tag{4.4}$$

- v_1 is an L^∞ weak solution of the transport equation

$$\begin{cases} \partial_t(\rho v_1) + \operatorname{div}(\rho \bar{v} v_1) = 0 \\ \\ v_1(0, \cdot) = -\frac{1}{\rho_+} \, . \end{cases} \tag{4.5}$$

Therefore, the vector field $\bar{v}$ is nearly incompressible in the sense of [**DL1**, Definition 3.6]. By the BV regularity of ρ and $\bar{v}$ we can apply Ambrosio's renormalization theorem [**DL1**, Theorem 4.1] and hence use [**DL1**, Lemma 5.10] to infer from (4.4) that the pair $(\rho, \bar{v})$ has the renormalization property of [**DL1**, Definition 3.9]. Thus we can apply [**DL1**, Corollary 3.14] to infer that there is a unique bounded weak solution of (4.5). Since the constant function is a solution, we therefore conclude that v_1 is identically equal to $-\frac{1}{\rho_+}$.

Set now $m(x_2, t) := \rho(x_2, t) v_2(x_2, t)$. The pair ρ, m is then a self-similar BV_{loc} weak solution of the 2×2 one-dimensional system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_{x_2} m = 0 \\ \\ \partial_t m + \partial_{x_2} \left(\frac{m^2}{\rho} + \rho^2 \right) = 0 \, , \end{cases} \tag{4.6}$$

that is the standard system of isentropic Euler in Eulerian coordinates with a particular polytropic pressure. It is well known that such system is genuinely nonlinear in the sense of [**Daf10**, Definition 7.5.1] and therefore, following the discusssion of [**Daf10**, Section 9.1] we conclude that the functions (ρ, m) result from "patching" rarefaction waves and shocks connecting constant states, i.e. they are classical solutions of the so-called Riemann problem in the sense of [**Daf10**, Section 9.3]. It is well known that in the special case of (4.6) the latter property and the admissibility condition determines uniquely the functions (ρ, m). For instance one can apply [**KK78**, Theorem 3.2]. □

4.2. Solution of the Riemann problem via wave curves

This section is devoted to an alternative way of proving uniqueness for the Riemann problem (4.1)–(4.2) which is based on the resolution of the initial jump discontinuity into wave funs. However, here we will study the case of Riemann data which allow for infinitely many solutions forward in time but are not necessarily generated by compression waves. In particular we choose the initial conditions of Section 3.5, i.e. $0 < \rho_- < \rho_+$ and $v_\pm = (\pm 1, 0)$ in (4.2). Moreover, we will treat pressure laws allowed by Lemma 3.5.2 in Chapter 3, which are essentially suitable smoothings of the step fucntion. Such pressures in turn do not satisfy the genuine nonlinearity condition $2p'(\rho) + \rho p''(\rho) > 0$; nonetheless uniqueness of self-similar solutions can be proven.

We first recall some basic facts which can be found in classical references as [**Daf10**] or [**Ser99**]. In order to study the Riemann problem for the isentropic compressible Euler system it is convenient to rewrite it in canonical form, i.e. in terms of the state variables (ρ, m) where m denotes the linear momentum, as done in Chapter 2:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(m) \;=\; 0 \\ \partial_t m + \operatorname{div}_x \left(\frac{m \otimes m}{\rho} \right) + \nabla_x p(\rho) \;=\; 0 \\ \rho(\cdot, 0) \;=\; \rho^0 \\ m(\cdot, 0) \;=\; m^0. \end{cases} \tag{4.7}$$

With the new variables, the entropy condition takes the following form:

$$\partial_t \left(\rho \varepsilon(\rho) + \frac{1}{2} \frac{|m|^2}{\rho} \right) + \operatorname{div}_x \left[\left(\varepsilon(\rho) + \frac{1}{2} \frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho} \right) m \right] \;\leq\; 0. \tag{4.8}$$

Similarly, we can rewrite the initial data (4.2) for (ρ, m) with the choices $0 < \rho_- < \rho_+$ and $v_\pm = (\pm 1, 0)$ in (4.2) so to obtain

$$(\rho^0(x), m^0(x)) := \begin{cases} (\rho_-, (-\rho_-, 0)) & \text{if } x_2 < 0 \\ \\ (\rho_+, (\rho_+, 0)) & \text{if } x_2 > 0, \end{cases} \tag{4.9}$$

where $\rho_\pm, v_\pm$ are constants. If we restrict our attention to pairs (ρ, m) which are admissible solutions of (4.7) and (4.9) and depend only on (x_2, t), then we will be dealing with a classical Riemann problem for (4.7) in one space-variable (only x_2) which admits self-similar solutions. Here we will show that, under the hypothesis of "self-similarity" of (ρ, m), such solutions are unique for some specific choices of the pressure and of the constants $\rho_\pm, v_\pm$. Surprisingly, Theorem 3.1.1 of Chapter 3 shows that uniqueness is completely lost if we drop the requirement that (ρ, m) depends only on $\frac{x_2}{t}$.

The plan is thus to prove uniqueness of weak admissible self–similar solutions (ρ, m) to the Riemann problem (4.7)-(4.9) which depend only on the space variable x_2:

$$(\rho(x, t), m(x, t)) = (\rho(x_2, t), m(x_2, t)).$$

Under our assumptions, it is convenient to make explicit the divergence operators in (4.7) and (4.8). Indeed, for $\rho(x,t) = \rho(x_2,t)$ and $m(x,t) = (m_1(x_2,t), m_2(x_2,t))$, we can write the system (4.7) as already done in Chapter 3 and obtain

$$\left\{\begin{array}{l} \partial_t \rho + \partial_{x_2}(m_2) \ = \ 0 \\ \partial_t m_1 + \partial_{x_2}\left(\frac{m_1 m_2}{\rho}\right) \ = \ 0 \\ \partial_t m_2 + \partial_{x_2}\left(\frac{{m_2}^2}{\rho} + p(\rho)\right) \ = \ 0 \\ \rho(\cdot,0) \ = \ \rho^0 \\ m(\cdot,0) \ = \ m^0, \end{array}\right. \tag{4.10}$$

while the energy inequality (4.8) becomes

$$\partial_t\left(\rho\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho}\right) + \partial_{x_2}\left[\left(\varepsilon(\rho) + \frac{1}{2}\frac{|m|^2}{\rho^2} + \frac{p(\rho)}{\rho}\right) m_2\right] \ \le \ 0. \tag{4.11}$$

If (ρ, m) is a self-similar solution of (4.10), focused at the origin, its restriction to $t > 0$ admits the representation

$$(\rho,m)(x,t) = (R,M)\left(\frac{x_2}{t}\right), \quad -\infty < x_2 < \infty, \quad 0 < t < \infty, \tag{4.12}$$

where (R, M) is a bounded measurable function on $(-\infty, \infty)$, which satisfies the ordinary differential equations

$$\begin{aligned} &[M_2(\xi) - \xi R(\xi)]^{\cdot} + R(\xi) = 0 \\ &\left[\frac{M_1(\xi)M_2(\xi)}{R(\xi)} - \xi M_1(\xi)\right]^{\cdot} + M_1(\xi) = 0 \\ &\left[\frac{M_2(\xi)^2}{R(\xi)} + p(R(\xi)) - \xi M_2(\xi)\right]^{\cdot} + M_2(\xi) = 0, \end{aligned}$$

in the sense of distributions.

Before discussing the resolution of the Riemann problem, we review some general features of system (4.10), which by the way have already been presented in Chapter 3 in the proof of Lemma 3.4.1. If we define the state vector $U := (\rho, m_1, m_2)$, we can recast the system (4.10) in the general form

$$\partial_t U + \partial_{x_2} F(U) = 0,$$

where

$$F(U) := \begin{pmatrix} m_2 \\ \frac{m_1 m_2}{\rho} \\ \frac{{m_2}^2}{\rho} + p(\rho) \end{pmatrix}.$$

By definition (cf. [**Daf10**]) the system (4.10) is hyperbolic since the Jacobian matrix $DF(U)$

$$DF(U) = \begin{pmatrix} 0 & 0 & 1 \\ \frac{-m_1 m_2}{\rho^2} & \frac{m_2}{\rho} & \frac{m_1}{\rho} \\ \frac{-{m_2}^2}{\rho^2} + p'(\rho) & 0 & \frac{2m_2}{\rho} \end{pmatrix}$$

has real eigenvalues

$$\lambda_1 = \frac{m_2}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2 = \frac{m_2}{\rho}, \qquad \lambda_3 = \frac{m_2}{\rho} + \sqrt{p'(\rho)} \tag{4.13}$$

and 3 linearly independent eigenvectors

$$R_1 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} - \sqrt{p'(\rho)} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 \\ \frac{m_1}{\rho} \\ \frac{m_2}{\rho} + \sqrt{p'(\rho)} \end{pmatrix}. \tag{4.14}$$

The eigenvalue λ_i of DF, $i = 1, 2, 3$, is called the *i-characteristic speed* of the system (4.10). On the part of the state space of our interest, with $\rho > 0$, the system (4.10) is indeed strictly hyperbolic. Finally, one can easily verify that the functions

$$w_3 = \frac{m_2}{\rho} + \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau, \qquad w_2 = \frac{m_1}{\rho}, \qquad w_1 = \frac{m_2}{\rho} - \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau \tag{4.15}$$

are, respectively, ($1-$ and $2-$), ($1-$ and $3-$), ($2-$ and $3-$) Riemann invariants of the system (4.10) (for the relevant definitions see [**Daf10**]). In other words there exist two $1-$Riemann invariants w_3 and w_2, two $2-$Riemann invariants w_1 and w_3 and two $3-$Riemann invariants w_2 and w_1.

We close this section with a key observation: note that the state variable m_1 appears only in the second equation of the system (4.10). We can thus "decouple" the study of the first and third equations in (4.10) from the study of the second one: this is possible by performing a sort of "projection" operation on the $\rho - m_2$–plane. In particular, we already know from the proof of Proposition 4.1.1 that the first component of the velocity field is uniquely determined by the second equation in (4.5) (since its initial value is yet bounded and hence [**DL1**, Corollary 3.14] still applies) and hence so is m_1 as soon as the density is.

Thus, we reduced ourselves to look for solutions (ρ, m_2) of the first and third equations in (4.10) and discuss their uniqueness only.

4.2.1. The Hugoniot locus. We focus our attention on the reduced system

$$\begin{cases} \partial_t \rho + \partial_{x_2}(m_2) \;=\; 0 \\ \partial_t m_2 + \partial_{x_2}\left(\frac{{m_2}^2}{\rho} + p(\rho)\right) \;=\; 0 \\ \rho(\cdot, 0) \;=\; \rho^0 \\ m_2(\cdot, 0) \;=\; (m^0)_2, \end{cases} \tag{4.16}$$

obtained by discarding the second equation in (4.10). Note that, if we define the state variable $\tilde{U} = (\rho, m_2)$ and we formally recast the system (4.16) in the form $\partial_t \tilde{U} + \partial_{x_2} G(\tilde{U}) = 0$ for $\tilde{U} = P_{1,3}U$, then we have $G(\tilde{U}) = P_{1,3}F(U)$ and $\widetilde{D}G(\tilde{U}) = P_{1,3}DF(U)P_{1,3}^T$ ($\widetilde{D} = D_{\tilde{U}}$), where $P_{1,3}$ is the following matrix:

$$P_{1,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, the characteristic speeds of the system (4.16) are λ_1 and λ_3 with associated eigenvectors $P_{1,3} \cdot R_1$ and $P_{1,3} \cdot R_3$ (see (4.13)-(4.14)). The *Hugoniot locus* of the reduced system (4.16) is the set of points $\tilde{U}_2 = (\rho, m_2)$ that may be joined to a fixed point $\tilde{U}_1 = (\overline{\rho}, \overline{m_2})$ by a shock. In view of the previous remark, we can observe that Hugoniot loci of the reduced system (4.16) correspond to projections on the $\rho - m_2$- plane of Hugoniot loci of the full system (4.10). In our case, we can describe the Hugoniot locus for (4.16) explicitly by computing the *Rankine-Hugoniot* jump conditions:

$$G(\tilde{U}_2) - G(\tilde{U}_1) = \sigma(\tilde{U}_2 - \tilde{U}_1),$$

which can be written, for U_1, U_2 such that $\tilde{U}_1 = P_{1,3}U_1$ and $\tilde{U}_2 = P_{1,3}U_2$, as

$$P_{1,3}[F(U_2) - F(U_1)] = P_{1,3}\sigma(U_2 - U_1).$$

Note, that the set of equations $[F(U_2) - F(U_1)] = \sigma(U_2 - U_1)$ describes indeed the Hugoniot locus of the full system (4.10).

Now, we would like to investigate the Hugoniot locus of the state (ρ_-, m_-). The state $(\rho_-, (m_-)_2) = (\rho_-, 0)$ on the left is joined to the state (ρ, m_2) on the right, by a shock of speed σ if the following equations hold:

$$\left\{\begin{array}{rcl} -\sigma(\rho-\rho_-)+m_2 &=& 0 \\ -\sigma m_2 + \frac{{m_2}^2}{\rho} + p(\rho) - p(\rho_-) &=& 0. \end{array}\right. \tag{4.17}$$

From (4.17) we infer that:

$$\sigma = \pm\sqrt{\frac{\rho(p(\rho)-p(\rho_-))}{\rho_-(\rho-\rho_-)}}. \tag{4.18}$$

Recalling the characteristic speeds λ_1 and λ_3 for the system (4.16), it is natural to call shocks propagating to the left $\left(\sigma_1 = -\sqrt{\frac{\rho(p(\rho)-p(\rho_-))}{\rho_-(\rho-\rho_-)}} < 0\right)$ 1-shocks and shocks propagating to the right $\left(\sigma_3 = \sqrt{\frac{\rho(p(\rho)-p(\rho_-))}{\rho_-(\rho-\rho_-)}} > 0\right)$ 3-shocks. Combining (4.17) with (4.18) we deduce that the Hugoniot locus of the point $(\rho_-, 0)$ in state space consists of two curves:

$$m_2 = \pm\sqrt{\frac{\rho(p(\rho)-p(\rho_-))}{\rho_-(\rho-\rho_-)}}\,(\rho-\rho_-)\,, \tag{4.19}$$

defined on the whole range of $\rho > 0$. Moreover a 1-shock joining $(\rho_-, 0)$ on the left to (ρ, m_2) on the right is admissible, i.e. it satisfies the entropy condition (4.8) if and only if $\rho_- < \rho$. While a 3-shock joining the state (ρ, m_2) on the left with the state $(\rho_+, 0)$ on the right is admissible if and only if $\rho > \rho_+$.

4.2.2. Rarefaction waves. In order to characterize rarefaction waves of the reduced system (4.16), we can refer to [**Daf10**, Theorem 7.6.6]: every i-Riemann invariant is constant along any i-rarefaction wave curve of the system (4.16) and conversely the i- rarefaction wave curve, through a state $(\overline{\rho}, \overline{m}_2)$ of genuine nonlinearity of the i-characteristic family, is determined implicitly by the system of equations $w_i(\rho, m_2) = w_i(\overline{\rho}, \overline{m}_2)$ for every $i-$Riemann invariant w_i. As an application of this Theorem, we obtain that the 1- and 3-rarefaction wave curves of the system (4.16) through the point $(\rho_-, 0)$ are determined respectively in terms of the Riemann invariants w_3 and w_1 by the equations

$$m_2 = \rho\int_{\rho}^{\rho_-}\frac{\sqrt{p'(\tau)}}{\tau}d\tau, \qquad m_2 = \rho\int_{\rho_-}^{\rho}\frac{\sqrt{p'(\tau)}}{\tau}d\tau. \tag{4.20}$$

The rarefaction waves through the point $(\rho_+, 0)$ can be obtained in a similar way.

4.3. Solution of the Riemann problem

According to [**Daf10**, Theorem 9.3.1] any self-similar solution of the Riemann problem (4.16), (4.9) with shocks satisfying the entropy inequality, comprises 3 constant states $U_0 = (\rho_0, m_0) = (\rho_-, m_-)$, U_1, $U_2 = (\rho_+, m_+)$. For $i = 1, 2$, U_{i-1} is joined to U_i by an i-wave.

In the following we will construct such a self-similar solution by piecing together shocks and rarefaction waves obtained in the previous sections. As in the literature (see for instance [**Daf10**]), we will call *forward* (or *backward*) *i-wave fan curve through* $(\overline{\rho}, \overline{m}_2)$ the Lipschitz curve $\Phi_i(\cdot, (\overline{\rho}, \overline{m}_2))$ (or $\Psi_i(\cdot, (\overline{\rho}, \overline{m}_2))$) describing the locus of states that may be joined on the right (or left) of the fixed state $(\overline{\rho}, \overline{m}_2)$ by an admissible i-wave fan. We consider the system (4.16) of two conservation laws in the two variables (ρ, m_2), then we will draw for it the forward 1-wave curve through the left state $(\rho_-, 0)$ and the backward 3-wave curve through the right state $(\rho_+, 0)$ and finally we will determine the intermediate state as the intersection of these two curves.

Recalling the form of the Hugoniot locus (4.19) and rarefaction wave curves (4.20) for system (4.16) with general pressure laws, we deduce that we can parametrize the wave curves employing ρ as the parameter. Thus, the forward 1-wave curve $m_2 = \Phi_1(\rho; (\rho_-, 0))$ through the point $(\rho_-, 0)$ consists of a 1-rarefaction wave for $\rho_- \geq \rho$ and an admissible 1-shock for $\rho_- < \rho$:

$$m_2 = \Phi_1(\rho; (\rho_-, 0)) = \begin{cases} \rho \int_\rho^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} d\tau & \text{if } \rho_- \geq \rho \\ -\sqrt{\frac{\rho(p(\rho)-p(\rho_-))}{\rho_-(\rho-\rho_-)}}\,(\rho - \rho_-) & \text{if } \rho_- < \rho. \end{cases} \tag{4.21}$$

On the other hand, the backward 3-wave curve through the point $(\rho_+, 0)$ is composed of a 3-rarefaction wave for $\rho_+ \geq \rho$ and an admissible 3-shock for $\rho_+ < \rho$:

$$m_2 = \Psi_3(\rho; (\rho_+, 0)) = \begin{cases} -\rho \int_\rho^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} d\tau & \text{if } \rho_+ \geq \rho \\ \sqrt{\frac{\rho(p(\rho)-p(\rho_+))}{\rho_+(\rho-\rho_+)}}\,(\rho - \rho_+) & \text{if } \rho_+ < \rho. \end{cases} \tag{4.22}$$

The intermediate constant state (ρ_M, m_M) is determined on the $\rho - m_2$ plane as the intersection of the forward 1-wave curve $\Phi_1(\rho; (\rho_-, 0))$ with the backward 3-wave curve $\Psi_3(\rho; (\rho_+, 0))$, namely by the equation

$$m_M = \Phi_1(\rho_M; (\rho_-, 0)) = \Psi_3(\rho_M; (\rho_+, 0))$$

(see Figure 1).

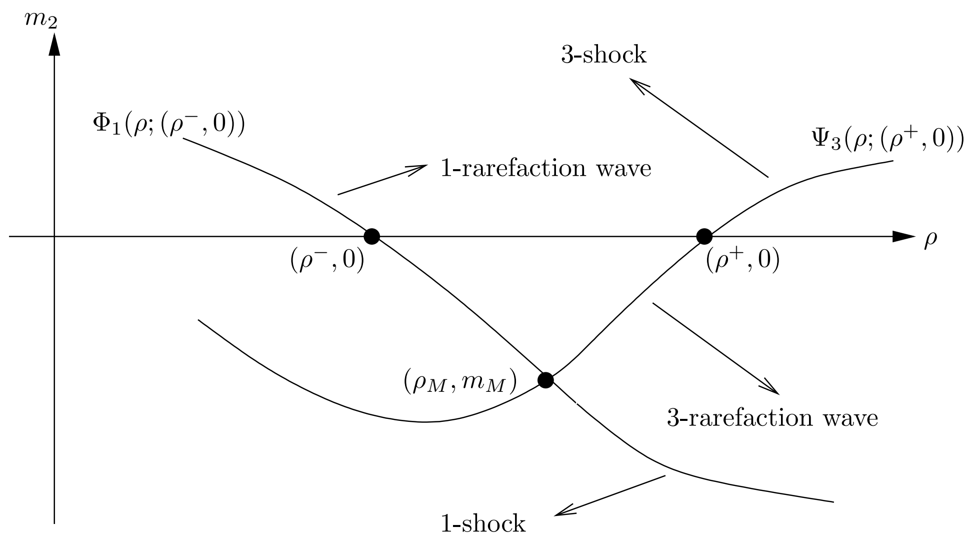

FIGURE 1. $\rho - m_2$ plane

We argue that such intersection is unique for the choice of pressure functions as in Lemma 3.5.2 of Chapter 3. Indeed m_M is uniquely determined, since Φ_1 is a strictly decreasing function in ρ for $\rho > \rho_-$ and $\Phi_1 > 0$ for $0 < \rho < \rho_-$, while Ψ_3 is positive for $\rho > \rho_+$ and negative for every $0 < \rho < \rho_+$ with Ψ_3 convex for $\rho < \widetilde{\rho}$ and concave for $\rho > \widetilde{\rho}$, where $\widetilde{\rho}$ is a density-value in a small neighborhood of 1 (recall from Section 3.5 of Chapter 3 that in the construction of the subsolution we chose $\overline{\rho} = 1$). Let us note that the second derivative of Ψ_3 with respect to ρ for $\rho < \rho_+$ is equal to $\frac{\rho p''(\rho)+2p'(\rho)}{2\rho\sqrt{p'(\rho)}}$; since $p' > 0$ the condition of convexity of Ψ_3 is exactly the genuine non–linearity condition $\rho p''(\rho) + 2p'(\rho) > 0$ which is not satisfied by our choice of pressure law.

The unique solution to the Riemann problem (4.16)-(4.9) for pressure laws as in Lemma 3.5.2, with end-states $(\rho_-, (m_-)_2)$ and $(\rho_+, (m_+)_2)$, comprises a compressive 1-shock joining $(\rho_-, (m_-)_2)$ with the state (ρ_M, m_M), followed by a 3-rarefaction wave, joining (ρ_M, m_M) with $(\rho_+, (m_+)_2)$.

CHAPTER 5

Existence of weak solutions

5.1. Introduction

The result presented in this Chapter stems from an idea recently explored by Emil Wiedemann for the incompressible Euler equations. In [**Wie11**] Wiedemann shows existence of weak solutions to the Cauchy problem for the incompressible Euler equations with general initial data (see Chapter 1). His proof combines some Fourier analysis with a clever application of the methods developed by De Lellis and Székelyhidi in [**DLS09**]-[**DLS10**] for the construction of non-standard solutions to the incompressible Euler equations. The conclusions achieved in [**Chi11**] and presented in Chapter 2 for the compressible Euler system gave hope that such an existence result could hold also for isentropic compressible gas dynamics in several space dimensions.

The existence of entropy solutions for the Cauchy problem associated with the isentropic compressible Euler equations in one space dimension was established, in the case of polytropic perfect gases first by DiPerna [**DP85**]–[**DP83**], Ding, Chen & Luo [**DCL85**], and Chen [**Che86**] based on compensated compactness arguments, and then, motivated by a kinetic formulation of hyperbolic conservation laws, by Lions, Perthame & Tadmor [**LPT94**], and Lions, Perthame & Souganidis [**LPS96**]. General pressure laws were covered first by Chen & LeFloch [**CL00**]. Unlike in the one-dimensional case, the existence problem for weak solutions of multi-dimensional isentropic gas dynamics has remained open so far.

The outcome of [**Wie11**] hints that the powerful approach by De Lellis and Székelyhidi is not only a "generator" of nonuniqueness, but can actually be exploited to construct weak solutions starting out from any initial data (see also [**DLS11**])! Here, we will follow such a hint and building upon results from [**Chi11**]-[**DLS10**]-[**Wie11**] we will show existence of weak solutions to the compressible Euler equations for any Lipschitz continuous initial density and any L^2 solenoidal initial momentum.

THEOREM 5.1.1. *Let $\rho^0 \in C^1_p(Q;\mathbb{R}^+)$ and $m^0 \in H(Q)$ such that* $\operatorname{div} m^0 = 0$. *Then there exists a global weak solution (ρ, m) (in fact, infinitely many) of the Cauchy problem for the compressible Euler equations with initial data (ρ^0, m^0).*

Of course, the optimal result would be existence of weak solutions starting out from any bounded initial data: Theorem 5.1.1 is a just a first step towards this.

5.2. The problem

In this section, we recall the isentropic compressible Euler equations of gas dynamics in n space dimensions, $n \geq 2$ (cf. Section 3.3 of [**Daf10**]) and in canonical form (as in Chapter 2). The system reads as

$$\begin{cases} \partial_t \rho + \operatorname{div}_x m \ = \ 0 \\ \partial_t m + \operatorname{div}_x \left(\frac{m\otimes m}{\rho}\right) + \nabla_x [p(\rho)] \ = \ 0 \\ \rho(\cdot, 0) \ = \ \rho^0 \\ m(\cdot, 0) \ = \ m^0 \end{cases}, \tag{5.1}$$

where ρ is the density and m the linear momentum field. The system is hyperbolic if the pressure p satisfies the following condition

$$p'(\rho) > 0.$$

We will consider here the case of general pressure laws given by a function p on $[0, \infty[$, that we always assume to be continuously differentiable on $[0, \infty[$ and strictly increasing on $[0, \infty[$.

Here, as in Chapter 2, we work with *space periodic* boundary conditions. For *space periodic* flows we assume that the fluid fills the entire space $\mathbb{R}^n$ but with the condition that m, ρ are periodic functions of the space variable. Let us recall the relevant definitions.

Let $Q = [0,1]^n$, $n \geq 2$ be the unit cube in $\mathbb{R}^n$. We denote by $H^m_p(Q)$, $m \in \mathbb{N}$, the space of functions which are in $H^m_{loc}(\mathbb{R}^n)$ and which are periodic with period Q:

$$f(x+l) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and every } l \in \mathbb{Z}^n.$$

For $m = 0$, $H^0_p(Q)$ coincides simply with $L^2(Q)$. Analogously, for every functional space X we define $X_p(Q)$ to be the space of functions which are locally (over $\mathbb{R}^n$) in X and are periodic of period Q. The functions in $H_p^m(Q)$ are easily characterized by their Fourier series expansion

(5.2)
$$H_p^m(Q) = \left\{ f \in L_p^2(Q) : \sum_{k\in\mathbb{Z}^n} |k|^{2m} \left|\widehat{f}(k)\right|^2 < \infty \text{ and } \widehat{f}(0) = 0 \right\},$$

where $\widehat{f} : \mathbb{Z}^n \to \mathbb{C}^n$ denotes the Fourier transform of f. We will use the notation $H(Q)$ for $H_p^0(Q)$ and $H_w(Q)$ for the space $H(Q)$ endowed with the weak L^2 topology.

By a *weak solution* of (5.1) on $\mathbb{R}^n \times [0,\infty[$ we mean a pair $(\rho, m) \in L^\infty([0,\infty[; L_p^\infty(Q))$ satisfying

(5.3)
$$|m(x,t)| \leq R\rho(x,t) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty[\text{ and some } R > 0,$$

and such that the following identities hold for every test functions $\psi \in C_c^\infty([0,\infty[; C_p^\infty(Q))$, $\phi \in C_c^\infty([0,\infty[; C_p^\infty(Q))$:

$$\int_0^\infty \int_Q [\rho\partial_t\psi + m\cdot\nabla_x\psi]\, dxdt + \int_Q \rho^0(x)\psi(x,0)dx \;=\; 0 \tag{5.4}$$

$$\begin{aligned} &\int_0^\infty \int_Q \left[m\cdot\partial_t\phi + \left\langle \frac{m\otimes m}{\rho}, \nabla_x\phi \right\rangle + p(\rho)\operatorname{div}_x\phi \right] dxdt \\ &+ \int_Q m^0(x)\cdot\phi(x,0)dx \;=\; 0. \end{aligned} \tag{5.5}$$

In the following, we will be dealing also with the semi-stationary Cauchy problem associated with the isentropic Euler equations:

$$\left\{ \begin{aligned} &\operatorname{div}_x m \;=\; 0 \\ &\partial_t m + \operatorname{div}_x\left(\tfrac{m\otimes m}{\rho}\right) + \nabla_x[p(\rho)] \;=\; 0 \\ &m(\cdot,0) \;=\; m^0. \end{aligned} \right. \tag{5.6}$$

A pair $(\rho, m) \in L_p^\infty(Q) \times L^\infty([0,\infty[; L_p^\infty(Q))$ is a *weak solution* on $\mathbb{R}^n \times [0,\infty[$ of (5.6) if $m(\cdot,t)$ is weakly-divergence free for almost every $0 < t < \infty$ and satisfies the following bound

(5.7)
$$|m(x,t)| \leq R\rho(x) \qquad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0,\infty[\text{ and some } R > 0,$$

and if the identity (5.5) holds for every $\phi \in C_c^\infty([0,\infty[; C_p^\infty(Q))$.

5.3. Existence of weak solutions

5.3.1. Background results. For the sake of completeness of the chapter, we report Proposition 4.1 from [**Chi11**] which is Proposition 2.4.1 in Chapter 2 and represents the building block of our argument. A similar criterion was proposed by De Lellis and Székelyhidi for the incompressible Euler equations (see [**DLS10**]) and used by Wiedemann in his proof of existence of weak solutions for incompressible Euler in [**Wie11**].

PROPOSITION 5.3.1. *Let $\overline{\rho} \in C^1_p(Q; \mathbb{R}^+)$ be any given density function.*
Assume there exist $(\overline{m}, \overline{U}, \overline{q})$ continuous space-periodic solutions of

$$\begin{aligned} \operatorname{div}_x \overline{m} &= 0 \\ \partial_t \overline{m} + \operatorname{div}_x \overline{U} + \nabla_x \overline{q} &= 0. \end{aligned} \tag{5.8}$$

on $\mathbb{R}^n \times]0, T[$ with

$$\overline{m} \in C([0, \infty]; H_w(Q)), \tag{5.9}$$

and a function $\chi \in C^\infty([0, T]; \mathbb{R}^+)$ such that

$$\lambda_{max}\left(\frac{\overline{m}(x,t) \otimes \overline{m}(x,t)}{\overline{\rho}(x)} - \overline{U}(x,t)\right) < \frac{\chi(t)}{n} \quad a.\ e.\ (x,t) \in \mathbb{R}^n \times]0,T[, \tag{5.10}$$

$$\overline{q}(x,t) = p(\overline{\rho}(x)) + \frac{\chi(t)}{n} \qquad for\ all\ (x,t) \in \mathbb{R}^n \times]0,T[. \tag{5.11}$$

Then there exist infinitely many weak solutions (ρ, m) of the system (5.6) in $\mathbb{R}^n \times [0, T[$ with density $\rho(x) = \overline{\rho}(x)$ and such that

$$m \in C([0, \infty]; H_w(Q)), \tag{5.12}$$

$$m(\cdot, t) = \overline{m}(\cdot, t) \qquad for\ t = 0, T\ and\ for\ a.e.\ x \in \mathbb{R}^n, \tag{5.13}$$

$$|m(x,t)|^2 = \overline{\rho}(x)\chi(t) \qquad for\ a.e.\ (x,t) \in \mathbb{R}^n \times]0,T[. \tag{5.14}$$

Let us remark that, according to the terminology used so far, a triple $(\overline{m}, \overline{U}, \overline{q})$ satisfying the hypothesis of Proposition 5.3.1 is called a *subsolution* of (5.6).

In the arguments of [**Chi11**], and hence of Chapter 2, the previous Proposition represents a criterion to recognize initial data m^0 allowing for many weak admissible solutions to (5.1). In our context it will play a different role: first, starting from any initial data (ρ^0, m^0) we will be able to construct a subsolution $(\overline{m}, \overline{U}, \overline{q})$ with the properties stated in the assumptions of Proposition 5.3.1 (with $\overline{\rho} := \rho^0$) and such that $\overline{m}(\cdot, 0) = m^0(\cdot)$, then it will be enough to apply Proposition 5.3.1 in order to prove existence of weak solutions (in fact, infinitely many) to the compressible Euler equations (5.1). Indeed, the solutions of (5.6) provided by Proposition 5.3.1 are also solutions of the full system (5.1).

5.3.2. Proof of Theorem 5.1.1. This section is devoted to the proof of Theorem 5.1.1, the main result of this note. For the sake of completeness we report here the statement.

THEOREM 5.3.2. *Let $\rho^0 \in C^1_p(Q; \mathbb{R}^+)$ and $m^0 \in H(Q)$ such that $\operatorname{div} m^0 = 0$. Then there exists a global weak solution (ρ, m) (in fact, infinitely many) of the Cauchy problem for the compressible Euler equations* (5.1).

Proof. The idea behind the proof is to choose suitably a subsolution $(\overline{m}, \overline{U}, \overline{q})$ satisfying the assumptions of Proposition 5.3.1 (with $\overline{\rho} := \rho^0$) and such that $\overline{m}(\cdot, 0) = m^0(\cdot)$, so that it will be enough to apply Proposition 5.3.1 in order to prove Theorem 5.1.1: indeed the conclusions of Proposition 5.3.1 and in particular (5.13) will yield our claim.

We first define via Fourier transform the following functions:

$$\widehat{\underline{m}}(k,t) = e^{-|k|t}\widehat{m}^0(k), \tag{5.15}$$

$$\widehat{\underline{U}}_{i,j}(k,t) = -i\left(\frac{k_j}{|k|}\widehat{\underline{m}}_i(k,t) + \frac{k_i}{|k|}\widehat{\underline{m}}_j(k,t)\right) \tag{5.16}$$

for every $k \neq 0$, and $\widehat{\underline{U}}(0,t) = 0$. Clearly, for $t > 0$, $\underline{m}$ and $\underline{U}$ are smooth. Moreover, $\underline{U}$ is symmetric and trace-free. The definition of $\widehat{\underline{m}}$ and $\widehat{\underline{U}}$ is taken from the construction of Wiedemann in [**Wie11**]. Let us note that the couple $(\widehat{\underline{m}}, \widehat{\underline{U}})$ defined by (5.15)-(5.16) satisfies the following system of equations in Fourier space:

$$\begin{cases} \partial_t \widehat{\underline{m}}_i + i\sum_{j=1}^n k_j \widehat{\underline{U}}_{i,j} = 0 \\ k \cdot \widehat{\underline{m}} = 0, \end{cases} \tag{5.17}$$

for $k \in \mathbb{Z}^n$, $i = 1, \ldots, n$. Hence $(\underline{m}, \underline{U})$ satisfies the system:

$$\begin{cases} \operatorname{div}_x \underline{m} \ = \ 0 \\ \partial_t \underline{m} + \operatorname{div}_x \underline{U} \ = \ 0. \end{cases} \tag{5.18}$$

Next, inspired by the proof of Proposition 7.1 in [**Chi11**], we define $\widetilde{U}$ componentwise by its Fourier transform as follows:

$$\widehat{\widetilde{U}}_{ij}(k) := \left(\frac{nk_ik_j}{(n-1)\,|k|^2}\right)\widehat{p(\rho^0(k))} \text{ if } i \neq j,$$

$$\widehat{\widetilde{U}}_{ii}(k) := \left(\frac{nk_i^2 - |k|^2}{(n-1)\,|k|^2}\right)\widehat{p(\rho^0(k))}. \tag{5.19}$$

for every $k \neq 0$, and $\widehat{\widetilde{U}}(0) = 0$. Also $\widetilde{U}$ thus defined is symmetric and trace-free. Moreover, since $p(\rho^0) \in C^1_p(\mathbb{R}^n)$, standard elliptic regularity arguments allow us to conclude that $\widetilde{U}$ is a continuous periodic matrix field. Next, notice that, by continuity of $\underline{m}$, ρ^0, $\underline{U}$ and $\widetilde{U}$, we have

$$\left\|\lambda_{max}\left(\frac{\underline{m}\otimes\underline{m}}{\rho^0} - \underline{U} - \widetilde{U}\right)\right\|_\infty = \widetilde{\lambda} \tag{5.20}$$

for some positive constant $\widetilde{\lambda}$. Therefore, we can choose any smooth function $\widetilde{\chi}$ on $\mathbb{R}$ such that $\widetilde{\chi} > n\widetilde{\lambda}$ on $[0,T]$ in order to ensure

$$\lambda_{max}\left(\frac{\underline{m}\otimes\underline{m}}{\rho^0} - \underline{U} - \widetilde{U}\right) < \frac{\widetilde{\chi}(t)}{n} \text{ for all } (x,t) \in \mathbb{R}^n \times [0,T[. \tag{5.21}$$

Now, let $\widetilde{q}$ be defined exactly as

$$\widetilde{q}(x,t) = p(\rho^0(x)) + \frac{\widetilde{\chi}(t)}{n} \text{ for all } x \in \mathbb{R}^n \times \mathbb{R} \tag{5.22}$$

for the choice of $\widetilde{\chi}$ just done. In light of (5.22), we can write the equation

$$\operatorname{div}_x \widetilde{U} + \nabla_x \widetilde{q} = 0 \qquad \text{on} \quad \mathbb{R}^n \times \mathbb{R} \tag{5.23}$$

in Fourier space as

$$\sum_{j=1}^n k_j \widehat{\widetilde{U}}_{ij} = k_i \widehat{p(\rho^0)} \tag{5.24}$$

for $k \in \mathbb{Z}^n$, $i = 1, \dots, n$. It is easy to check that $\widehat{\widetilde{U}}$ as defined by (5.19) solves (5.24) and hence $\widetilde{U}$ and $\widetilde{q}$ satisfy (5.23).

Now, given $\overline{\rho} := \rho^0$, we are ready to choose $(\overline{m}, \overline{U}, \overline{q})$. We set:

$$\overline{m}(x,t) := \underline{m}(x,t), \tag{5.25}$$

$$\overline{U}(x,t) := \underline{U}(x,t) + \widetilde{U}(x), \tag{5.26}$$

$$\overline{q}(x,t) := \widetilde{q}(x,t). \tag{5.27}$$

It remains to show that the subsolution defined by (5.25)-(5.26)-(5.27) satisfies the assumptions of Proposition 5.3.1.

First, we notice that system (5.8) is trivially satisfied by $(\overline{m}, \overline{U}, \overline{q})$ as a consequence of (5.18) and (5.23). Finally, with the choice $\chi := \widetilde{\chi}$, the subsolution $(\overline{m}, \overline{U}, \overline{q})$ will satisfy also (5.10)-(5.11) thanks to the definition of $\widetilde{q}$ in (5.22) and to the property (5.21) of $\widetilde{\chi}$. Since T can be chosen to be $+\infty$, by Proposition 5.3.1 we find infinitely many solutions $m \in C([0,\infty[; H_w(Q))$ of (5.6) on $\mathbb{R}^n \times [0,\infty[$ with density ρ^0. Now, define $\rho(x,t) = \rho_0(x)\mathbf{1}_{[0,\infty[}(t)$. This shows that (5.5) holds. To prove (5.4) observe that ρ is independent of t and m is weakly divergence-free for almost every $0 < t < \infty$. Therefore, the pair (ρ, m) is a global weak solution of (5.1) with initial data (ρ^0, m^0) as desired. $\square$

Bibliography

[Amb04] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, **158** (2):227–260, 2004.

[ABDL04] L. Ambrosio, F. Bouchut, and C. De Lellis. Well–posedness for a class of hyperbolic systems of conservation laws in several space dimensions. *Comm. PDE*, **29** (9–10):1635–1651, 2004.

[ADL03] L. Ambrosio and C. De Lellis. Existence of solutions for a class of hyperbolic systems of conservation laws in several space dimensions. *Int. Math. Res. Not.*, (41):2205–2220, 2003.

[BB05] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math.*, **161** (2):1–120, 2005.

[BDLS11] Y. Brenier, C. De Lellis, and L. Szekelyhidi, Jr. Weak-strong uniqueness for measure valued solutions. *Comm. Math. Phys.*, **305** (2):351–361, 2011.

[Bre66] P. Brenner. The Cauchy problem for symmetric hyperbolic systems in L^p. *Math. Scand.*, **19** (1):27–37, 1966.

[Bre95] A. Bressan. The unique limit of the Glimm scheme. *Arch. Rational Mech. Anal.*, **130**:205–230, 1995.

[Bre00] A. Bressan. *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000.

[BF94] A. Bressan and F. Flores. On total differential inclusions. *Rend. Sem. Mat. Univ. Padova*, **92**:9–16, 1994.

[BDS13] T. Buckmaster T., C. De Lellis and L.J. Székelyhidi. Transporting microstructure and dissipative Euler flows. *Preprint*, 2013.

[Che86] G.-Q. Chen. Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III). *Acta Math. Sci.*, **6** (1):75–120, 1986.

[CL00] G.-Q. Chen and P.G. LeFloch. Compressible Euler equations with general pressure law. *Arch. Rational Mech. Anal.*, **153** (3):221–259, 2000.

[Chi11] E. Chiodaroli. A counterexample to well-posedeness of entropy solutions to the compressible Euler system. *Preprint*, 2011.

[CDL] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill–posedness of the isentropic system of gas dynamics. *Preprint*, 2013.

[CDLK] E. Chiodaroli, C. De Lellis, and O. Kreml. Convex integration for the compressible Euler system. *In preparation*.

[Cho12] A. Choffrut. h-principles for the incompressible Euler equations. *Preprint*, 2012.

[CS96] E.D. Conway and J.A. Smoller. Global solutions to the Cauchy problem for quasi-linear first order equations in several space variables. *Comm. Pure Appl. Math.*, **19** (1):95–105, 1996.

[CFG09] D. Cordoba, D. Faraco, and F. Gancedo. Lack of uniqueness for weak solutions of the incompressible porous media equation. *Arch. Ration. Mech. Anal.*, **200** (3): 725–746, 2011.

[DM97] B. Dacorogna and P. Marcellini. General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial cases. *Acta Math.*, **178** (1):1–37, 1997.

[Daf10] C.M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundleheren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Third edition Springer, Berlin, 2010.

[DA13] S. Daneri. Cauchy problem for dissipative Hölder solutions to the incompressible Euler equations. *Preprint*, 2013.

[DL1] C. De Lellis. Notes on hyperbolic systems of conservation laws and transport equations. *Handbook of differential equations: evolutionary equations*, volume III: 277–382, 2007.

[DLOW03] C. De Lellis, F. Otto, and M. Westdickenberg. Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Rational Mech. Anal.*, **170** (2) :137–184, 2003.

[DLR03] C. De Lellis and T. Riviére. The rectifiability of entropy measures in one space dimension. . *J. Math. Pures Appl.*, **82** (10):1343–1367, 2003.

[DLS09] C. De Lellis and L.J. Székelyhidi. The Euler equations as a differential inclusion. *Ann. Math.*, **170** (3):1417–1436, 2009.

[DLS10] C. De Lellis and L.J. Székelyhidi. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, **195** (1):225–260, 2010.

[DLS11] C. De Lellis and L.J. Székelyhidi. The h-principle and the equations of fluid dynamics. *Bull. Amer. Math. Soc. (N.S.)*, **195** (3):347–375, 2012.

[DLS12] C. De Lellis and L.J. Székelyhidi. Continuous dissipative Euler flows. *Preprint*, 2012. To appear in *Inv. Math.*

[DLSZ12] C. De Lellis and L.J. Székelyhidi. Dissipative Euler flows and Onsager's conjecture. *Preprint*, 2012.

[DCL85] X. Ding, G.-Q. Chen, and P. Luo. Convergence of the Lax-Friedrichs scheme for the isentropic gas dynamics (I)(II). *Acta Math. Sci.*, **5**:483–500, 501–540, 1985.

[DP83] R. J. Di Perna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, **91** (1):1–30, 1983.

[DP85] R. J. Di Perna. Compensated compactness and general systems of conservation laws. *Trans. Amer. Math. Soc.*, **292** (2):383–420, 1985.

[DR00] J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity 13*, **13** (1):249–255, 2000.

[EM02] Y Eliashberg and N. Mishaev. *Introduction to the h-principle*, volume 48 of *Graduate studies in Mathematics*. American Mathematical Society, Providence, 2002.

[Ell06] V. Elling. A possible counterexample to well posedeness of entropy solutions and to Godunov scheme convergence. *Math. Comp.*, **75** (256):1721–1733, 2006.

[Fil67] A. F. Filippov. Classical solutions of differential equations with multi-valued right-hand side. *SIAM J. Control*, **5** (4):609–621, 1967.

[Gli65] J. Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, **18** (4):697–715, 1965.

[Gro70] M. Gromov. A topological technique for the construction of solutions of differential equations and inequalities. *ICM*, **2**:221–225, 1970.

[Gro86] M. Gromov. *Partial differential relations*, volume 3 of *Ergebnisse der Mathematik*. Springer, 1986.

[Hug89] H. Hugoniot. Sur la propagation du movement dans le corps et spécialement dans les gaz parfaits, I;II. *J. Ecole Polytechnique*, **57**:3–97, 1887, **58**:1–125, 1889.

[KK78] B.L. Keyfitz, and H.C. Kranzer. Existence and uniqueness of entropy solutions to the Riemann problem for hyperbolic systems of two nonlinear conservation laws. *J.* Differential Equations, **27** (4):444–476, 1978.

[Kir03] B. Kirchheim. Rigidity and geometry of microstructures. Habilitation thesis, University of Leipzig, 2003.

[KMS03] B. Kirchheim, S. Müller, and V. Sverák. Studying nonlinear PDE by geometry in matrix space. In *Geometric analysis and nonlinear partial differential equations*, 347–395. Springer, 2003.

[Kru70] S. Kruzkov. First-order quasilinear equations with several space variables. *Mat. Sbornik*, **123**:228–255, 1970.

[Kui95] N.H. Kuiper. On C^1 isometric embeddings, I. *Nederl. Akad. Wetensch. Proc.*, A **58**:545–556, 1995.

[Lax57] P.D. Lax. Hyperbolic systems of conservation laws. *Comm. Pure Appl. Math.*, **10** (4):537–566, 1957.

[Lax71] P.D. Lax. Shock waves and entropy. In *Contributions to Functional Analysis*, 603–634. E. A. Zarantonello, New York: Academic Press, 1971.

[Lio96] P.L. Lions. *Mathematical topics in fluid mechanics*, volume 3 of *Oxford Lecture Series in Mathemtics and its Applications*. Oxford Science Publications, Oxford, 1996.

[LPS96] P.-L. Lions, B. Perthame, and P.E. Souganidis. Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in eulerian and lagrangian coordinates. *Comm. Pure Appl. Math.*, **49** (6):599–638, 1996.

[LPT94] P.-L. Lions, B. Perthame, and E. Tadmor. Kinetic formulation for the isentropic gas dynamics and p-systems. *Commun. Math. Phys.*, **163** (2):415–431, 1994.

[MS03] S. Müller and V. Sverák. Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. Math.*, **157** (3):715–742, 2003.

[Nas54] J. Nash. C^1 isometric embeddings. *Ann. Math.*, **60** (3):383–396, 1954.

[Ole54] O.A. Oleinik. The Cauchy problem for nonlinear equations in a class of discontinuous functions. *Dokl. Akad. Nauk SSSR*, **95**:451–454, 1954.

[Ole57] O.A. Oleinik. Discontinuous solutions of non-linear differential equations. *Usp. Mat. Nauk*, **12**:3–73, 1957.

[Ole59] O.A. Oleinik. Uniqueness and stability of the generalized solution of the Cauchy problem for quasilinear equation. *Usp. Mat. Nauk*, **14**:165–170, 1959.

[Oxt90] J.C. Oxtoby. *Measure and category*, volume 2 of *Graduate Texts in Mathematics*. Springer-Verlag, 1990.

[Ran70] W.J.M. Rankine. On the thermodynamic theory of waves of finite longitudinal disturbance. *Phil. Trans. Royal Soc. London*, **160**:277–288, 1870.

[Rau86] J. Rauch. BV estimates fail for most quasilinear systems in dimension greater than one. *Comm. Math. Phys.*, **106** (3):481–484, 1986.

[Ray10] Lord Rayleigh. Aerial plane waves of finite amplitude. *Proc. Royal Soc. London*, **84A**:247–284, 1910.

[Sch93] V. Scheffer. An inviscid flow with compact support in space-time. *J. Geom. Anal.*, **3** (4):343–401, 1993.

[Ser99] D. Serre. *Systems of conservation laws. 1.*, volume 3. Cambridge University Press, Cambridge, 1999.

[Shn97] A. Shnirelman. On the nonuniqueness of weak solution of the Euler equation. *Comm. Pure Appl. Math.*, **50** (12):1261–1286, 1997.

[Shv11] R. Shvydkoy. Convex integration for a class of active scalar equations. *J. Amer. Math. Soc.*, **24** (4):1159–1174, 2011.

[Sma58] S. Smale. A classification of immersions of the two-sphere. *Trans. Amer. Math. Soc.*, **90**:281–290, 1959.

[Sma59] S. Smale. The classification of immersions of spheres in euclidean spaces. *Ann. Math.*, **69** (2):327–344, 1959.

[Sto48] G.G. Stokes. On a difficulty in the theory of sound. *Philos. Magazine*, Ser. 3, **33**:349–356, 1848.

[Sz11] L.J. Székelyhidi. Weak solutions to the incompressible Euler equations with vortex sheet initial data. *C. R. Acad. Sci. Paris*, Ser. I, **349** (19-20):1063–1066, 2011.

[Szé11] L.J. Székelyhidi. Relaxation of the incompressible porous medium equation. *Preprint*, 2011. To appear in *Ann. l'ENS.*

[SW11] L.J. Székelyhidi and E. Wiedemann. Young measures generated by ideal incompressible fluid flows. *Arch. Ration. Mech. Anal.* **206** (1):333-366, 2012.

[Syc01] M. A. Sychev. Comparing two methods of resolving homogeneous differential inclusions. *Calc. Var.*, **13** (22):213–229, 2001.

[Tar79] L. Tartar. Compensated compactness and applications to partial differential equations. *Nonlinear analysis and mechanics: Heriot-Watt Symposium* volume IV:136–212, R.J. Knops, London: Pitman, 1979.

[Vol67] A.I. Volpert. The spaces BV and quasilinear equations. *Mat. Sbornik*, **73**:255–302, 1967.

[Wie11] E. Wiedemann. Existence of weak solutions for the incompressible Euler equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire (C)*, **28** (5): 727–730, 2011.